# BIQUOTIENTS WITH SINGLY GENERATED RATIONAL COHOMOLOGY

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Abstract. We classify all biquotients whose rational cohomology rings are generated by one element. As a consequence we show that the Gromoll-Meyer 7-sphere is the only exotic sphere which can be written as a biquotient.

Let $G$ be a compact Lie group and $H \subset G \times G$ be a compact subgroup. Then $H$ acts on $G$ on the left by the formula $(h_1, h_2)g = h_1 g h_2^{-1}$. If this action happens to be free the orbit space is a manifold which is called a *biquotient* of $G$ by $H$ and denoted by $G/\!/H$. In the special case when $H$ has the form $K_1 \times K_2$ where $K_1 \subset G \times 1$ and $K_2 \subset 1 \times G$ we will often write $K_1 \backslash G / K_2$ instead of $G/\!/(K_1 \times K_2)$.

Biquotients are natural generalizations of homogeneous spaces, and like homogeneous spaces, have metrics with nonnegative sectional curvature induced by biinvariant metrics on $G$. The concept of a biqoutient was first introduced by Gromoll-Meyer in [GM], where they showed that one of these biquotients, $Sp(2)/\!/Sp(1)$, is an exotic 7-sphere, which produced the first example of an exotic sphere with nonnegative curvature. Biquotients were later on examined more systematically in [Es],[Bo] in the context of a search for new manifolds with positive sectional curvature. In fact, all known examples of manifolds admitting metrics of positive sectional curvature are given by biquotients.

Some attempts were made to find other exotic spheres which could be written as biquotients but they proved unsuccessful. We will show in this paper that any such attempt must indeed fail. More generally we classify the biquotients which are rationally spheres and projective spaces, extending a well known classification in the homogeneous case [Be, p.195-196]:

Theorem A. *Let $M = G/\!/H$ be a compact, simply connected biquotient whose rational cohomology ring is generated by one element. Then $M$ is either diffeomorphic to a compact rank one symmetric space, or it is diffeomorphic to one of the eight homogeneous spaces or four biquotients in Table B.*

Some comments may be helpful, in order to understand the examples in this Table. The subscript for the 3 dimensional subgroups denotes the index of the subgroup, where a simple subgroup $H$ in a simple Lie group $G$ has index $k$ if the induced map $\pi_3(H) \simeq \mathbb{Z} \to \pi_3(G) \simeq \mathbb{Z}$ is multiplication by $\pm k$, which in particular means that $\pi_3(G/H) \simeq \mathbb{Z}_k$. Notice that $Sp(1)_{10}$ is the unique maximal 3 dimensional subgroup in $Sp(2)$, such that $Sp(2)/Sp(1)_{10}$ is the normal homogeneous Berger space with positive curvature (in fact the only entry in Table B which is known to admit a metric with positive sectional curvature).

The 3 dimensional subgroups in $G_2$ can be described as follows: In $G_2$ one has the equal rank subgroups $SO(4)$ and $SU(3)$. The subgroup $SO(4)$ contains two normal $SU(2)$'s. One of them has index one in $G_2$ and is also contained in $SU(3) \subset G_2$. The quotient $G_2/SU(2)_1$ is diffeomorphic to $SO(7)/SO(5)$. The other $SU(2) \subset SO(4)$ has index 3 in $G_2$. One also has

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| $G//H$ | range of $n$ | rational type |
|---|---|---|
| $SO(2n+1)/(SO(2n-1)\times SO(2))$ | $n\geq 2$ | $\mathbb{CP}^{2n-1}$ |
| $SO(2n+1)/SO(2n-1)$ | $n\geq 2$ | $\mathbb{S}^{4n-1}$ |
| $SU(3)/SO(3)_2$ | | $\mathbb{S}^5$ |
| $Sp(2)/Sp(1)_{10}$ | | $\mathbb{S}^7$ |
| $G_2/SU(2)_3$ | | $\mathbb{S}^{11}$ |
| $G_2/SO(3)_4$ | | $\mathbb{S}^{11}$ |
| $G_2/SO(3)_{28}$ | | $\mathbb{S}^{11}$ |
| $G_2/SO(4)$ | | $\mathbb{HP}^2$ |
| $\triangle SO(2)\backslash SO(2n+1)/SO(2n-1)$ | $n\geq 2$ | $\mathbb{CP}^{2n-1}$ |
| $\triangle SU(2)\backslash SO(4n+1)/SO(4n-1)$ | $n\geq 2$ | $\mathbb{HP}^{2n-1}$ |
| $Sp(2)//Sp(1)$ | | $\mathbb{S}^7$ |
| $G_2//SU(2)$ | | $\mathbb{S}^{11}$ |

TABLE B. Rational Spheres and Projective Spaces

a subgroup $SO(3)\subset SO(4)$ which has index 4 in $G_2$, and is also contained in $SU(3)$. Finally there exists a maximal $SO(3)$ in $G_2$ which has index 28.

The biquotient $G_2//SU(2)$ is obtained by letting $SU(2)$ act via the index three $SU(2)$ on the left, and the index four $SO(3)$ on the right. The Gromoll-Meyer sphere $Sp(2)//Sp(1)$ is obtained by letting $Sp(1)$ act via $\mathrm{diag}(q,q)$ on the left, and $\mathrm{diag}(q,1)$ on the right. In the two even dimensional biquotients, the subgroup on the left is embedded as $\mathrm{diag}(1,A,\cdots,A)$ where $A$ lies either in $SO(2)$ or in $SU(2)$. Of these four biquotients, all but the Gromoll-Meyer sphere were first discovered by Eschenburg in [Es], except that he did not discuss their topological properties.

By computing the cohomology rings and the Pontrjagin classes, we will show that none of these spaces are homeomorphic to each other, except that $Sp(2)//Sp(1)$ is homeomorphic to $S^7$ [GM], and the rational 11 sphere $G_2//SU(2)$ is homeomorphic to $SO(7)/SO(5)\simeq T^1S^6\simeq G_2/SU(2)$. At the moment we are unable to decide if the last two spaces are diffeomorphic or not, but we at least show they can only differ from each other by a connected sum with one of the 992 homotopy 11-spheres.

In particular we obtain the following

COROLLARY C. *The only biquotient which can be an exotic sphere is diffeomorphic to the Gromoll-Meyer sphere $Sp(2)//Sp(1)$.*

Some of the other spaces given by Theorem A also have interesting relationships. We will show that the Grassmannian $SO(2n+1)/(SO(2n-1)\times SO(2))$ and the biquotient $\triangle SO(2)\backslash SO(2n+1)/SO(2n-1)$ have the same cohomology rings, and they also have the same integral cohomology groups as $\mathbb{CP}^{2n-1}$, but they can be distinguished by their Pontrjagin classes. Similarly, $\triangle SU(2)\backslash SO(4n+1)/SO(4n-1)$ has the same integral cohomology groups as $\mathbb{HP}^{2n-1}$, but a different ring structure.

After a first version of this paper was finished, the preprint [T] by B.Totaro came to our attention, where the author independently classifies all biquotients which are rational homology spheres. At the same time B.Wilking pointed out to us that the reduction to the simple case, which in our first version was more complicated, can be easily achieved by Lemma 1.4, which we then noticed is also Lemma 4.5 in [T]. We will use this simplified version of the proof. We would finally like to thank B. Wilking for several further useful comments.

## 1. Reduction to the case of a simple $G$

Throughout this section all cohomology have rational coefficients and all homotopy groups are tensored with $\mathbb{Q}$. Also for the purposes of the proof we will use the following equivalent but formally stronger definition of a biquotient, see [Es]:

Let $H \xrightarrow{\rho} G \times G$ be a homomorphism and let $\Delta Z_G$ be the diagonal embedding of the center of $G$ into $G \times G$. Let $Z = \rho^{-1}(\Delta Z_G)$. It is clear that $Z$ lies in the kernel of the usually defined biquotient action of $H$ on $G$. Suppose the biquotient action of $H/Z$ on $G$ is free. Then the quotient space is a manifold diffeomorphic to $(G/\rho(Z))//(H/Z)$ which we will still denote by $G//H$.

In this section we will also not distinguish between a simple group and its various covers, using e.g. the same notation for $SO(n)$ and $Spin(n)$.

We first need to recall some well known facts about rational cohomology of Lie groups.

A Lie group of rank $n$ is rationally homotopy equivalent to a product of a finitely many odd dimensional spheres $S_1^{2k_1+1} \times ... \times S_n^{2k_n+1}$ . The dimensions of the spheres corresponding to various simple groups are listed in Table 1.1

| $G$ | $\dim S_i$ |
|---|---|
| $SO(2n-1)$ | $3, 7, ..., 4n-5$ |
| $SO(2n)$ | $3, 7, ..., 4n-5, 2n-1$ |
| $SU(n)$ | $3, 5, ..., 2n-1$ |
| $Sp(n)$ | $3, 7, ..., 4n-1$ |
| $G_2$ | $3, 11$ |
| $F_4$ | $3, 11, 15, 23$ |
| $E_6$ | $3, 9, 11, 15, 17, 23$ |
| $E_7$ | $3, 11, 15, 19, 23, 27, 35$ |
| $E_8$ | $3, 15, 23, 27, 35, 39, 47, 59$ |

Table 1.1. Dimensions of Spheres

Proposition 1.2. *Suppose $M = G//H$ is a biquotient such that $M$ is simply connected and the cohomology algebra $H^*(M, \mathbb{Q})$ is generated by one element. Then there exists a biquotient $G'//H'$ such that $G'$ is simple and $M$ is diffeomorphic to $G'//H'$.*

*Proof.* The key in the proof of Proposition 1.2 is the following elementary Lemma the proof of which is left to the reader.

Lemma 1.3. *Let $G$ be a compact Lie group acting differentiably on manifolds $X$ and $Y$. Suppose that the action of $G$ on $X$ is transitive and the diagonal action of $G$ on $X \times Y$ is free. Then for any $x \in X$ the action of isotropy group $G_x$ on $Y$ is free and the quotient spaces $(X \times Y)/G$ and $Y/G_x$ are canonically diffeomorphic. Moreover, if the action of $G$ on $X \times Y$ is a biquotient action then the action of $G_x$ on $Y$ is again a biquotient action.*

First notice that by passing to a finite cover we can assume that both $G$ and $H$ are products of compact simple or abelian groups. Indeed, let $\pi : G' \to G$ be a finite cover of $G$ that splits as a product of simple or abelian groups $G' = G_1 \times ... \times G_n$. Let $\hat{H} = \pi^{-1}(H)$. Then $G'//\hat{H} \simeq G//H$ and since $M$ is simply connected, $\hat{H}$ is connected. Let $H' \to \hat{H}$ be a finite cover such that $H'$ splits as $H' = H_1 \times .. \times H_m$. Then $G'//H' \simeq G'//\hat{H} \simeq G//H$. From now on we can assume that $G$ and $H$ already have the product forms $G = G_1 \times ... \times G_n, H = H_1 \times .. \times H_m$. Furthermore, since $M$ is simply connected, by Lemma 1.3 we can assume that $G$ has no abelian factors.

Next let us describe the rational homotopy type of $M$. We are given that $H^*(M, \mathbb{Q})$ is generated by one element $a$, which easily implies that $M$ is formal. Indeed, the naturally defined map $(H^*(M, \mathbb{R}), 0) \to (\Omega^*(M), d)$ is clearly a DGA quasi-isomorphism. Thus $M$ is formal over $\mathbb{R}$ and hence, by the field extension theorem [FHT, page 156], it is formal over $\mathbb{Q}$.

If $\deg a$ is odd, it is obvious that $\dim M = \deg a$ and $M$ is rationally equivalent to $S^{\deg a}$. If $\deg a = 2k$ is even, $H^*(M) = \mathbb{Q}[a]/a^{m+1}$ where $m = \dim M/\deg a$. By formality, the minimal model of $M$ is then the same as the minimal model of $(\mathbb{Q}[a]/a^{m+1}, 0)$ and is equal to $(\mathbb{Q}[x, y], d)$ where $\deg x = \deg a, \deg y = (m + 1)\deg a - 1, dx = 0, dy = x^{m+1}$. In particular, $M$ has exactly two nontrivial rational homotopy groups $\pi_{\deg a}(M) \simeq \pi_{\deg y}(M) \simeq \mathbb{Q}$. In either case $M$ has exactly one nontrivial odd homotopy group.

Let $i = (i_1, \dots, i_n) : H \to G^2 = G_1^2 \times \dots G_n^2$ be the fiber inclusion.

By looking at the long exact homotopy sequence of the fibration $H \xrightarrow{i} G \to M$ we see that the induced map $i_* : \pi_*(H) \to \pi_*(G)$ satisfies $\dim \operatorname{coker}(i_*) = 1$ and $\dim \ker(i_*) = 1 (= 0)$ if $\dim M$ is even (odd). Since $\operatorname{rank}(G) = \dim \pi_*(G)$ and $\operatorname{rank}(H) = \dim \pi_*(H)$ this implies that $\operatorname{rank}(G) = \operatorname{rank}(H)$ if $\dim M$ is even and $\operatorname{rank}(G) = \operatorname{rank}(H) + 1$ if $\dim M$ is odd.

By the above, all but one of the coordinate projections $i_k : H \to G_k$ are onto on $\pi_*$.

Lemma 1.4. *Let $f : H \to G$ be a continuous map between compact connected Lie groups. Suppose the induced map $f_* : \pi_*(H) \to \pi_*(G)$ is onto. Then $f$ is onto.*

*Proof.* We are going to show that the induced map $f^* : H^*(G) \to H^*(H)$ is injective. First observe that since both $H$ and $G$ are rationally products of odd-dimensional spheres their cohomology algebras are free exterior algebras on a finite number of odd-dimensional generators. Thus for both $H$ and $G$ the vector spaces spanned by those generators (denoted by $V_H$ and $V_G$ respectively) can be naturally identified with quotients of $H^*$ by decomposable elements $H^{*+}/(H^{*+} \cdot H^{*+})$. The assumptions of the Lemma implies that the induced map $f^* : H^{*+}(G)/(H^{*+}(G) \cdot H^{*+}(G)) \to H^{*+}(H)/(H^{*+}(H) \cdot H^{*+}(H))$ is injective. Since $H^*(H) \simeq \Lambda V_H$ this implies that the map $f^* : H^*(G) \to H^*(H)$ is injective. In particular, the image of the fundamental cohomology class $[G]$ is nonzero and hence $f$ is onto. □

By Lemma 1.4 for all but one factor $G_i$ the action of $H$ on $G_i$ is transitive. Therefore by Lemma 1.3 we can reduce the number of simple factors of $G$ to one. This concludes the proof of Proposition 1.2.

□

## 2. Case of a simple $G$ and Proof of Theorem A

We are now ready to proceed with the proof of Theorem A in the Introduction. We can assume that $H^*(M) = \mathbb{Q}[a]/a^{m+1}$, and that $M = G/\!/H$ with $G$ simple.

Let the embedding $H \subset G\times G$ be given by $(j^-, j^+)$ where $j^-$ and $j^+$ are two homomorphisms. If one of these is trivial, we are in the situation of a homogeneous space where we can use the classification in [Be, p.195-196] or [On] to obtain the first half of Table B. By the proof of Proposition 1.2 we have that $\operatorname{rank} G = \operatorname{rank} H$ if $\dim M$ is even and $\operatorname{rank} G = \operatorname{rank} H + 1$ if $\dim M$ is odd. Also recall that for any simple Lie group $\operatorname{rank} \pi_3 = 1$. Therefore $H$ is simple if $\deg a > 4$, $H$ has two simple factors if $\deg a = 4$, $H = H_1 \times S^1$ with $H_1$ simple if $\deg a = 2$ and $H$ is trivial if $\deg a = 3$.

We will now distinguish between the case $\dim M$ odd and $\dim M$ even.

If $\dim M$ is odd and hence $H^*(M) = H^*(S^{2n+1})$, the proof of Proposition 1.2 shows the following: If $\dim M = 3$, by above $H$ must be trivial and hence $G/\!/H$ is homogeneous. If $\dim M > 3$, then $G$ and $H$ are both simple, $\operatorname{rank} G = \operatorname{rank} H + 1$, and $H^*(G) \cong H^*(H \times S^{2n+1})$ as rings. Hence $j^-$ and $j^+$ are either homomorphisms with finite kernels, or trivial. Now one can easily produce a list of all simple pairs $H \subset G$ such that $H^*(G) \cong H^*(H \times S^{2n+1})$, using Table 1.1 and elementary representation theory. The result is summarized in Table 2.1. Notice that this happens to agree with the list of homogeneous spaces $G/H$ which are odd dimensional rational homology spheres (see [Be, p.195-196] and [On]), although this is not a priori clear.

| $G$ | $H$ | range of $n$ | number of reps |
|---|---|---|---|
| $SO(2n)$ | $SO(2n-1)$ | $n \geq 3, n \neq 4$ | 1 |
| $SU(n)$ | $SU(n-1)$ | $n \geq 4$ | 1 |
| $Sp(n)$ | $Sp(n-1)$ | $n \geq 3$ | 1 |
| $SO(2n+1)$ | $SO(2n-1)$ | $n \geq 3$ | 1 |
| $Spin(7)$ | $G_2$ | | 1 |
| $Spin(8)$ | $Spin(7)$ | | 3 |
| $SU(3)$ | $SU(2)$ | | 2 |
| $Sp(2)$ | $Sp(1)$ | | 3 |
| $G_2$ | $SU(2)$ | | 4 |

Table 2.1. Rational odd dimensional homology spheres

In the first 5 cases, the embedding of $H$ in $G$ is unique up to conjugacy and hence these cases only give rise to homogeneous biquotients. In the remaining cases there exist at least two embeddings of $H$ and hence the possibility of a biquotient.

The three representations of $Spin(7)$ in $Spin(8)$ intersect in $G_2$ and hence this case cannot give rise to a biquotient. The group $SU(3)$ has the index 1 subgroup $SU(2)$ and the index 2 subgroup $SO(3)$ which intersect in a circle and hence cannot give rise to a biquotient. The group $Sp(2)$ has the index one subgroup $Sp(1) \times 1 \subset Sp(1) \times Sp(1) \subset Sp(2)$, the index 2 subgroup $\triangle Sp(1) \subset Sp(1) \times Sp(1) \subset Sp(2)$, and the maximal index 28 subgroup $Sp(1) \subset Sp(2)$. It is not hard to see that only the first two can be combined to give rise to a biquotient, the Gromoll Meyer sphere $Sp(2)/\!/Sp(1)$. The exceptional group $G_2$ has 4 three dimensional subgroups described in the Introduction. The question which biquotients this gives rise to is more complicated. However, the general situation of $\operatorname{rank} G = 2$ and $\operatorname{rank} H = 1$ has been completely examined in [Es, p.166-170] where it was shown that it gives rise to only two biquotients. The first one is the Gromoll Meyer sphere and the second one is $G_2//SU(2)$, where one uses the index 3 and index 4 subgroups for $j^-$ and $j+$.

If $\dim M$ is even, as explained above we have $\operatorname{rank} G = \operatorname{rank} H$ where $G$ is simple and $H$ has at most two simple factors. If $G//H$ is not homogeneous, the maximal torus in $H$ must give rise to a (two-sided) biquotient action of a torus on a simple Lie group $G$, whose dimension is equal to the rank of $G$. These were all classified in [Es]. Such biquotient tori actions are fairly rare, and in particular none exist for the exceptional Lie groups. Furthermore for each such maximal torus action Eschenburg determines the maximal extension to a group $K$ of the same rank as the torus, which hence automatically also acts freely on $G$. These maximal extensions are listed in [Es, Table 101], although, as was pointed out in [EKS], one entry was left out. For reader's convenience, we reproduce this corrected list in Table 2.2.

It turns out that they are all of the form $G//(K_1 \times K_2)$ with $\operatorname{rank} K_1 = 1$ and $K_2$ semisimple such that $K_2$ is embedded only on one side. For our purpose, $H$ does not have to be maximal, however we have that $H \subset K_1 \times K_2$ is a subgroup of equal rank and thus it must be of the form $H_1 \times H_2$ with $H_i \subset K_i$ and $\operatorname{rank} H_i = \operatorname{rank} K_i$.

By the above both $H_1$ and $H_2$ must be simple and therefore, we only need to consider those cases where $H_1 = K_1$ or $H_1 = S^1 \subset K_1$ and $H_2 \subset K_2$ with both $H_2$ and $K_2$ simple of equal rank. The only possibilities for $H_2 \neq K_2$ we need to consider are $SO(2n) \subset SO(2n+1)$ and $SU(3) \subset G_2$ since the list of all possible pairs of simple groups with simple subgroups of equal rank is well known and quite short [Wo, p.281], and since in Eschenburg's Table the only exceptional group that occur for $K_2$ is $G_2$.

Therefore there are very few examples that we have to add to Table 2.2 to be considered for our even dimensional biquotients. It is then easy to see that the only ones whose rational cohomology is generated by one element and which might not be diffeomorphic to a homogeneous space are the two entries $\triangle SO(2)\backslash SO(2n+1)/SO(2n-1)$ and $\triangle SU(2)\backslash SO(4n+1)/SO(4n-1)$ in Table B. Here $\triangle SO(2)$ and $\triangle SU(2)$ stand for "Hopf actions" $\operatorname{diag}(1, A, \cdots, A)$ where $A$ lies either in $SO(2)$ or in $SU(2)$. Notice though that in Table 2.2 there are quite a few biquotients which are diffeomorphic to $\mathbb{C}P^n$ or $\mathbb{H}P^n$ without being homogeneous. This finishes the proof of Theorem A.

## 3. Diffeomorphism classification

Theorem 3.1. *None of the spaces listed in Theorem A are mutually diffeomorphic except possibly the rational* 11*-spheres* $G_2//SU(2)$ *and* $SO(7)/SO(5)$ *(which can also be written as* $G_2/SU(2)_1$*). These two spaces are PL-homeomorphic but may possibly differ by a connected sum with an exotic* 11*-sphere.*

*Proof.* The homogeneous spaces can easily be differentiated from the rank one symmetric spaces and from each other by the torsion in their cohomology, see e.g. [MZ].

Here we will only need the integral cohomology groups of $SO(2n+1)/SO(2n-1) = T^1S^{2n}$, which follows easily from the Gysin sequence of the bundle $S^{2n-1} \to T^1S^{2n} \to S^{2n}$ :

$$H^*(T^1S^{2n}) = \begin{cases} \mathbb{Z} \text{ if } * = 0, 4n-1 \\ Z_2 \text{ if } * = 2n \\ 0 \text{ otherwise} \end{cases}$$

| $G$ | restrictions | $K_1$ | $K_2$ | $M$ |
|---|---|---|---|---|
| $SU(n)$ | $n \geq 3, 1 \leq k < \frac{n}{2}$ | $S^1_{\mp,k}$ | $SU(n-1)$ | $\mathbb{C}P^{n-1}$ |
| $SO(2n)$ | $n = 2m+1 \geq 3$ | $\triangle SO(2)$ | $SO(2n-1)$ | $\mathbb{C}P^{n-1}$ |
| $SU(n)$ | $n = 2m \geq 4$ | $\triangle SU(2)$ | $SU(n-1)$ | $\mathbb{H}P^{m-1}$ |
| $SO(2n)$ | $n = 2m \geq 4$ | $\triangle SU(2)$ | $SO(2n-1)$ | $\mathbb{H}P^{m-1}$ |
| $Sp(n)$ | $n \geq 2$ | $\triangle Sp(1)$ | $Sp(n-1)$ | $\mathbb{H}P^{n-1}$ |
| $Spin(9)$ | | $\triangle Sp(1)$ | $Spin(7)$ | $\mathbb{H}P^3$ |
| $Spin(8)$ | | $\triangle Sp(1)$ | $Spin(7)'$ | $S^4$ |
| $Spin(7)$ | | $\triangle Sp(1)$ | $G_2$ | $S^4$ |
| $SU(n)$ | $n \geq 5, 2 \leq k < \frac{n}{2}$ | $S^1_{\mp}$ | $SU(k)SU(n-k)$ | |
| $SU(n)$ | $n = 2m \geq 4$ | $S^1$ | $SU(m)SU(m)$ | |
| $SO(2n+1)$ | $n = 2m-1 \geq 3$ | $\triangle SO(2)$ | $SO(2n-1)$ | |
| $SO(2n)$ | $n \geq 3, 3 \leq p \leq n,\ n, p$ odd | $S^1$ | $SO(2n-p)SO(p)$ | |
| $SO(2n)$ | $n \geq 5$ | $SU(2)$ | $SU(n)$ | |
| $SO(2n+1)$ | $n = 2m \geq 4$ | $\triangle SU(2)$ | $SO(2n-1)$ | |
| $SO(2n+1)$ | $n = 2m-1 \geq 3$ | $\triangle SO(2)$ | $SO(2n-1)$ | |
| $SO(2n)$ | $n \geq 8, 3 \leq p < n,\ n$ even $,p$ odd | $SU(2)$ | $SO(2n-p)SO(p)$ | |
| $Sp(n)$ | $n \geq 3$ | $Sp(1)$ | $SU(n)$ | |
| $Sp(4)$ | | $SU(2)$ | $SU(2)SU(2)SU(2)$ | |

TABLE 2.2. Maximal biquotients of equal rank

Let us next consider the rational $\mathbb{CP}^{2n-1}$'s $M = \triangle SO(2)\backslash SO(2n+1)/SO(2n-1)$ and $N = SO(2n+1/SO(2n-1) \times SO(2), n > 1$.

Let us first compute the integral cohomology rings of $M$ and $N$. From the Gysin sequence of the bundle $S^1 \to T^1S^{2n} \to M$ we compute

$$H^*(M) = \begin{cases} \mathbb{Z} \text{ if } * = 2k, \text{ for } k = 0, ..., 2n-1 \\ 0 \text{ otherwise} \end{cases}$$

Moreover from the same sequence we see that the Euler class of this bundle $a_M \in H^2(M)$ is a generator of $H^2(M)$ and the following sequences are exact

$$0 \to H^{2k-2}(M) \overset{\cup a_M}{\to} H^{2k}(M) \to 0 \text{ for } k = 1, ..., n-1, n+1, ..., 2n-1$$
$$0 \to H^{2n-2}(M) \overset{\cup a_M}{\to} H^{2n}(M) \to Z_2 \to 0$$

Hence the ring structure is determined by the fact that $a_M^n$ is twice a generator in $H^{2n}(M)$ and thus $M$ is not homotopy equivalent to $\mathbb{C}P^{2n-1}$.

The same argument works for $N$ and thus $M$ and $N$ have isomorphic cohomology rings. To compare $M$ and $N$ to each other we will show that they have different rational Pontrjagin classes and thus are not homeomorphic.

Observe that both $M$ and $N$ are quotients of $T^1(S^{2n})$ by different free $S^1$ actions. Let us describe these actions explicitly. We will identify $T^1(S^{2n})$ with the set $\{(x,y)\in\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}||x|=|y|=1,\langle x,y\rangle=0\}$. By construction, the $S^1$ action producing $M$ is the diagonal action $z(x,y)=(z(x),z(y))$ for the embedding $S^1\to SO(2n)\to SO(2n+1)$ with the first embedding given by the Hopf action. Observe that this action leaves the product $S^{2n}\times S^{2n}$ invariant. It is easy to see that the normal bundle $\nu$ of $T^1(S^{2n})$ inside $\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}$ is trivial. Consider the natural orthonormal trivialization $e: T^1(S^{2n})\times\mathbb{R}^3\to\nu$ given by $e_1(x,y)=e((x,y),(1,0,0))=(x,0), e_2(x,y)=e((x,y),(0,1,0))=(0,y), e_3(x,y)=e((x,y),(0,0,1))=\frac{1}{\sqrt{2}}(y,x)$. It is easy to see that with respect to $e$ the action of $S^1$ on $\mathbb{R}^3$ is trivial and therefore $\nu$ descends to a trivial bundle over $M$.

Let $p: T^1(S^{2n})\to M$ be the canonical projection. Then $TT^1(S^{2n})\simeq p^{\#}(TM)\oplus T_F$ where $T_F$ is the tangent bundle to the fiber. It is obvious that $T_F\simeq\epsilon^1$ is a trivial bundle over $T^1(S^{2n})$ and therefore

$$TT^1(S^{2n})\simeq p^{\#}(TM)\oplus\epsilon^1$$

Next note that the action of $S^1$ on $\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}$ is equivalent to the sum of $2n$ copies of the standard representation and a 2-dimensional trivial representation $\epsilon^2$.

Combining the previous formulas we obtain the following identity

$$TM\oplus\epsilon^4\simeq 2n\gamma_M\oplus\epsilon^2 \tag{3.2}$$

where $\gamma_M$ is the rank-two bundle over $M$ associated to the principal $S^1$ bundle $T^1(S^{2n})\to M$ and the canonical $S^1\simeq SO(2)$ action on $\mathbb{R}^2$. By construction, $e(\gamma_M)=a_M$, the generator of $H^2(M)\simeq\mathbb{Z}$.

Therefore,

$$p_1(TM)=p_1(TM\oplus\epsilon^4)=p_1(2n\gamma_M)=2np_1(\gamma_M)=2ne(\gamma_M)^2=2na_M^2 \tag{3.3}$$

Let us now compute the first Pontrjagin class of $N$. By definition, the $S^1$ action on $T^1(S^{2n})$ which produces $N$ is given by the following formula:

$$e^{it}(x,y)=(\cos tx+\sin ty,-\sin tx+\cos ty)$$

In other words this is just the geodesic flow action for the round metric on $S^{2n}$.

As before we see that it is equivalent to the sum of $2n+1$ copies of the standard representation and therefore it descends to the bundle $(2n+1)\gamma_N$ over $N$.

On the other hand, by the same argument as before we see that

$$(2n+1)\gamma\simeq TN\oplus\bar\nu\oplus\epsilon^1 \tag{3.4}$$

where $\bar\nu$ is the $S^1$ quotient of the normal bundle $\nu$ to $T^1(S^{2n})$ inside $\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}$. Let us study $\bar\nu$ further. As was discussed earlier, $\nu$ is a trivial bundle. It is easy to see that with respect to the trivialization $e=(e_1,e_2,e_3)$ the action $\rho$ of $S^1$ on $\mathbb{R}^3$ corresponding to $\nu$ is given by the following matrix

$$e^{it}\longrightarrow\begin{pmatrix}\cos^2 t & \sin^2 t & \sqrt{2}\sin t\cos t\\ \sin^2 t & \cos^2 t & -\sqrt{2}\sin t\cos t\\ -\sqrt{2}\sin t\cos t & \sqrt{2}\sin t\cos t & \cos^2 t-\sin^2 t\end{pmatrix} \tag{3.5}$$

From formula 3.5 we see that $\rho$ is equivalent to the sum of a rank-one trivial representation and a representation of weight 2. Therefore it descends to the representation $\eta \oplus \epsilon^1$ where $\eta_N$ is a rank-two bundle over $N$ with Euler class $2a_N$. We can now rewrite formula 3.4 as follows

$$(2n+1)\gamma \simeq TN \oplus \eta_N \oplus \epsilon^2 \tag{3.6}$$

Therefore $(2n+1)a_N^2 = (2n+1)p_1(\gamma_N) = p_1(TN) + p_1(\eta_N) = p_1(TN) + e(\eta_N)^2 = p_1(TN) + 4a_N^2$ and hence

$$p_1(TN) = (2n-3)a_N^2 \tag{3.7}$$

Finally, observe that the groups $H^4(M)/\langle p_1(M)\rangle$ and $H^4(N)/\langle p_1(N)\rangle$ are cyclic. By comparing (3.3) and (3.7) we see that these groups have different orders and therefore $M$ and $N$ are not homeomorphic by topological invariance of rational Pontrjagin classes.

Next let us consider the rational $\mathbb{HP}^{2n-1}$ given by $M = \triangle SU(2)\backslash SO(4n+1)/SO(4n-1)$. A similar computation to the one in case of rational $\mathbb{CP}^n$'s shows that it has the following cohomology

$$H^*(M) = \begin{cases} \mathbb{Z} \text{ if } * = 4k, \text{ for } k = 0, ..., 2n-1 \\ 0 \text{ otherwise} \end{cases}$$

Also, as before, if $a_M \in H^4(M)$ is a generator of $H^4(M)$ then the following sequences are exact

$$0 \to H^{4k-4}(M) \stackrel{\cup a_M}{\to} H^{2k}(M) \to 0 \text{ for } k = 1, ..., n-1, n+1, ..., 2n-1$$
$$0 \to H^{4n-4}(M) \stackrel{\cup a_M}{\to} H^{4n}(M) \to Z_2 \to 0$$

Therefore $a_M^n$ is twice a generator in $H^{4n}(M)$ and hence $M$ is not homotopy equivalent and hence not diffeomorphic to $\mathbb{HP}^{2n-1}$.

The biquotient $Sp(2)//Sp(1)$ is homeomorphic but not diffeomorphic to $\mathbb{S}^7$ according to [GM].

Let us finally discuss the rational 11-sphere $M^{11} = G_2//SU(2)$. Recall that

$$H^*(G_2) = \begin{cases} \mathbb{Z} \text{ if } * = 0, 3, 11, 14 \\ Z_2 \text{ if } * = 6, 9 \\ 0 \text{ otherwise} \end{cases}$$

Since the fiber in the fibration $SU(2) \to G_2//SU(2)$ is given by the composition of two maps $(j^-, j^+) : SU(2) \hookrightarrow G_2 \times G_2$ and $\times : G_2 \times G_2 \to G_2$, where $\times(g_1, g_2) = g_1 \cdot g_2^{-1}$, it induces the map $j_*^- - j_*^+$ in $\pi_3$. Since $j^-$ is given by the index 3 subgroup and $j^+$ by the index 4 subgroup, it follows that the fiber inclusion $SU(2) \to G_2$ is an isomorphism on $\pi_3$. ¿From the long exact homotopy sequence of the fibration $S^3 = SU(2) \to G_2 \to M$ we conclude that $M$ is 4-connected. Therefore the Euler class $e \in H^4(M)$ of this bundle is zero. From the Gysin sequence

$$\to H^1(M) \stackrel{\cup e}{\to} H^5(M) \to H^5(G_2) \to$$

we see that $H^5(M) = 0$. Similarly, from

$$\to H^2(M) \stackrel{\cup e}{\to} H^6(M) \to H^6(G_2) \to H^3(M) \to$$

we see that $H^6(M) \simeq H^6(G_2) \simeq \mathbb{Z}_2$. Thus

$$H^*(M) = \begin{cases} \mathbb{Z} \text{ if } * = 0, 11 \\ Z_2 \text{ if } * = 6 \\ 0 \text{ otherwise} \end{cases}$$

and by Poincare duality

$$H_*(M) = \begin{cases} \mathbb{Z} \text{ if } * = 0, 11 \\ Z_2 \text{ if } * = 5 \\ 0 \text{ otherwise} \end{cases} \tag{3.8}$$

We will say that two closed manifolds are almost diffeomorphic if they differ by a connected sum with a homotopy sphere.

LEMMA 3.9. *Suppose $X^{11}$ is a simply-connected smooth manifold with homology given by (3.8). Then $X$ is almost diffeomorphic to $T^1S^6$.*

*Proof.* The almost diffeomorphism classification of $k$-connected $2k+1$ manifolds with $k \neq 3, 7$ was carried out by Wall [Wa]. By assumptions, our manifold is as above with $k = 5$.

According to Wall, the *oriented* almost diffeomorphism class of a 4-connected 11-manifold is completely determined by the following set of invariants:

- $G = H_5(X)$;
- A nonsingular bilinear form (called the linking form) $b : G^* \times G^* \to G^*$where $G^*$ is the torsion subgroup of $G$;
- A quadratic form $q : G^* \to \mathbb{Q}/2\mathbb{Z}$ associated with the bilinear form $2b$;
- A homomorphism $\alpha : G \to \pi_4(SO)$.

In our case $G \simeq G^* \simeq \mathbb{Z}_2$. Since there exists only one non-degenerate bilinear form on $\mathbb{Z}_2$, the form $b$ is uniquely determined.

By Bott periodicity, $\pi_4(SO) = 0$, and thus $\alpha = 0$. The only remaining Wall invariant which has to be determined is the quadratic form $q$. To compute it we need to recall its definition.

Look at the generator $a$ in $H_5(X) = Z_2$ given by a map $a : S^5 \to X$. By Whitney's theorem we can assume that $a$ is an embedding. The normal bundle to $a$ is trivial since $\pi_4(SO(6)) = 0$.

Choose a section $a_1$ of the normal bundle to $a$ such that the normal bundle to $a_1$ in the unit tangent bundle of $a$ is trivial. This is not automatic since $\pi_4(SO(5)) = Z_2$ . The easiest way to achieve this is to take the obvious section corresponding to any trivialization of the normal bundle. Let $a_2 : S^5 \to M$ be the normal sphere in the unit tangent bundle. The orientation on $a_2$ is uniquely determined by the orientations on $M$ and $a$. More explicitly, we orient the normal $D^6$ to have intersection with $a$ equal to $+1$ and consider the induced orientation on the normal $S^5 = \partial D^6$. Let $Y = X \backslash a(S^5)$ and let $y_1 = [a_1], y_2 = [a_2]$ be the homology classes in $X$ given by $\alpha_i$. Then it can be shown that $y_2$ generates the kernel of the map $H_5(Y) \to H_5(X)$ which is infinite cyclic. It is clear that $2y_1$ lies in that kernel and therefore $2y_1 = \lambda y_2$. It can be shown [Wa] that the quotient $\lambda/2$ is well-defined mod $2\mathbb{Z}$ and we set $q(a) :\stackrel{\text{def}}{=} \lambda/2 \bmod 2$. It is obvious that $\lambda$ can not be even so $q(a)$ can only take values $\pm 1/2 \bmod 2$.

If we change the orientation of $X$ then, by construction, $y_2$ changes to $-y_2$ and hence $q$ changes to $-q$. Therefore, $X$ and $-X$ ( which stands for the same manifold with opposite orientation) have different oriented almost diffeomorphism types and any other oriented manifold satisfying the assumptions of the Lemma (e.g. $T^1S^6$) is orientably almost diffeomorphic to either $X$ or $-X$. □

*Remark* 3.10. Observe that two 11-manifolds satisfying Lemma 3.9 are almost diffeomorphic iff they are $PL$-equivalent. Indeed, a manifold $X$ satisfying Lemma 3.9 is homeomorphic to $T^1S^6$. In particular it admits a $CW$ decomposition $e^0 \cup e^5 \cup e^6 \cup e^{11}$. Look at the universal bundle $PL/O \to B_O \to B_{PL}$. We wish to classify different smooth structures inside a fixed $PL$ structure on $X$, i.e we have to classify the homotopy types of all possible lifts of the classifying map $f : X \to B_{PL}$. By the general obstruction theory, the obstruction $o_i$ to extend a homotopy between two lifts from the $i-1$'th to the $i$'th skeleton of $X$ lives in $H^i(X, \pi_i(O/PL))$. Since $O/PL$ is 6-connected we have $o_i = 0$ for $i = 1, \dots, 6$. The CW structure of $X$ then implies that $o_i = 0$ for $i = 7, \dots, 10$. Thus the only possible nontrivial obstruction is $o_{11}$ and the PL class of $X$ contains at most $|H^{11}(X, \pi_{11}(O/PL))| = |\pi_{11}(O/PL)| = 992$ distinct diffeomorphism types. On the other hand, connected sums of $X$ with different homotopy spheres have different Eells-Kuiper invariants [EK] and thus are non-diffeomorphic. Since there are exactly $|\pi_{11}(O/PL)| = 992$ homotopy 11-spheres, the conclusion follows.

Thus the oriented diffeomorphism type of $M$ is determined by its oriented $PL$-homeomorphism type together with the Eells-Kuiper invariant of $M$ which at the moment we are unable to compute.

$\square$

## References

[Be] A. Besse, *Manifolds all of whose geodesics are closed*, Springer Verlag 1978.

[Bo] R. Bock, *Doppelquotienten ungerader dimension und positive Schnittkrümmung*, Ph.D thesis, Augsburg 1999.

[D1] E. B. Dynkin, *The maximal subgroups of the classical groups*, Transl. Am. Math. Soc. Series 2, Vol. 6 (1957), 245-378.

[D2] E. B. Dynkin, *Semisimpe subalgebras of semisimple Lie algebras*, Transl. Am. Math. Soc. Series 2, Vol. 6 (1957), 111-244.

[EK] J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Ann. Mat. Pura Appl. (4) **60** (1962), 93–110.

[FHT] Y. Félix , S. Halperin, and J.-C. Thomas , *Rational homotopy theory*, Springr Verlag 2001.

[GM] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. (2) **100** (1974), 401–406.

[Es] J. Eschenburg, *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, Schriften der Math. Universität Münster **32** (2), 1984.

[EKS] J. Eschenburg , A. Kollross and K. Shankar, *Free isometric circle actions on compact symmetric spaces*, to appear in Geometriae Dedicata.

[MZ] J. McCleary and W. Ziller, *On the free loop space of homogeneous spaces*, Am.J.Math. 109 (1987), 765-782.

[On] A. L. Oniscik, *Transitive compact transformation groups*, Amer. Math. Soc. Transl. **55** (1966), 153–194.

[T] B. Totaro, *Cheeger manifolds and the classification of biquotients*, Preprint 2002.

[Wa] C.T.C. Wall,*Classification problems in differential topology. VI. Classification of* $(s-1)$*-connected* $(2s+1)$*-manifolds*, Topology 6 (1967), 273–296.

[Wo] J.A. Wolf, *Spaces of Constant Curvature*, Publish or Perish Inc., 1977.

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