SHARP BOUNDS AND RIGIDITY FOR VOLUMES OF BOUNDARIES OF ALEXANDROV SPACES

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ABSTRACT. We obtain sharp volume bounds on the boundaries of Alexandrov spaces with given lower curvature bound, dimension, and radius. We also completely classify the rigidity case and analyze almost rigidity. Our results are new even for smooth manifolds with locally convex boundary.

CONTENTS

1. INTRODUCTION

Let (X^m, d) be an Alexandrov space with curvature bounded below by some $\kappa \in \mathbb{R}$ and dimension m. Recall that the radius $\mathrm{Rad}(X)$ of X is defined as

$$\mathrm{Rad}(X) = \inf_{x \in X} \sup_{y \in X} \mathrm{d}(x, y).$$

We denote by $M^m(\kappa)$ the m-dimensional model space with constant curvature κ and π_κ the diameter of $M^m(\kappa)$, i.e., $\pi_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$ and ∞ otherwise.

Our first result is a sharp upper bound on the volume of the boundary of X of radius R.

Theorem 1.1. Let X^m be Alexandrov of curv $\geq \kappa$. Let $R > 0$ and assume in addition that $R \leq \pi_\kappa/2$ if $\kappa > 0$. If $\overline{B_R(p)} = X$ for some $p \in X$ then

$$(1.2) \qquad\qquad \mathcal{H}^{m-1}(\partial X) \leq \mathcal{H}^{m-1}(\partial B_R^\kappa),$$

where B_R^κ is the ball of radius R in $M^m(\kappa)$ (we drop the dependence on m for simplicity since we will fix dimension). Furthermore, if $\overline{B_R(p)} = X$ for some $p \in \partial X$ then

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$$(1.3) \qquad\qquad \mathcal{H}^{m-1}(\partial X) \leq \mathcal{H}^{m-1}(\partial B_{R,+}^{\kappa}),$$

where $B_{R,+}^{\kappa}$ is the half-ball of radius R in $M^m(\kappa)$, i.e., it is the ball of radius R around a boundary point in the half space $M_+^m(\kappa)$ in $M^m(\kappa)$.

Remark 1.4. Note that $\mathcal{H}^{m-1}(\partial B_{R,+}^{\kappa}) < \mathcal{H}^{m-1}(\partial B_R^{\kappa})$ unless $\kappa > 0$ and $R = \pi_\kappa/2$ in which case they are equal. Hence, except for that case the upper bound given by Theorem 1.1 is stronger if $p \in \partial X$.

Remark 1.5. The bound given by Theorem 1.1 is new even in the smooth setting, i.e., when $X = (M^m, g)$ is a compact smooth Riemannian manifold of sec $\geq \kappa$ with locally convex boundary.

In the special case $\kappa > 0, R = \pi_\kappa/2$, the estimate given by Theorem 1.1 follows from Petrunin's solution to Lytchak's problem, who proved in [Pet07][Section 3.3.5] that if X^m is Alexandrov of curv ≥ 1 then $\mathcal{H}^{m-1}(\partial X) \leq \mathrm{vol}_{m-1} \mathbb{S}^{m-1}$. Note that if X^m is Alexandrov of curv ≥ 1 with nonempty boundary and $p \in X$ is the soul of X then $X = B_{\pi/2}(p)$, see [Pet07][Section 3.3.5] or [Per91, Section 6.3]. Therefore Petrunin's result gives the sharp upper bound in Theorem 1.1 for $\kappa > 0, R = \pi_\kappa/2$. However, Petrunin's proof does not generalize to other values of R and κ.

Next, let us note that an implicit bound on the volume of the boundary in this theorem follows from a result of Fujioka [Fuj18, Theorem 6.5], where an upper bound is obtained on the volume of any extremal subset of X in terms of dimension, lower curvature and upper diameter bounds of X. Fujioka's result also gives an implicit local bound under the assumptions of Theorem 1.9 below.

As was pointed out to the authors by Daniele Semola, an implicit upper bound on $\mathcal{H}^{m-1}(\partial X)$ under the assumptions of Theorems 1.1 and 1.9 also follows from work of Li and Naber [LN20]. Since this is not mentioned explicitly in their paper let us elaborate on this point. In [LN20][Corollary 1.4] it is shown that for any integer m and $\varepsilon > 0, \kappa \in \mathbb{R}$, there is $C = C(m, \varepsilon, \kappa) > 0$ such that if X^m Alexandrov space of curv $\geq \kappa$ and $p \in X$ then

$$(1.6) \qquad\qquad \mathcal{H}^k(\mathcal{S}_\varepsilon^k(X) \cap B_1(p)) \leq C.$$

Here $\mathcal{S}_\varepsilon^k(X)$ stands for the (k, ε) quantitative stratified set. See [LN20] for the definition. For any small $\varepsilon < \varepsilon_0(m, \kappa)$ it is immediate from the definition that if $k = m-1$ and $p \notin \mathcal{S}_\varepsilon^{m-1}(X)$ then p is (m, ε)-strained and hence it cannot be a boundary point by [BP92]. Therefore $\partial X \subset \mathcal{S}_\varepsilon^{m-1}(X)$ for any $\varepsilon < \varepsilon_0$ and so (1.6) gives a uniform bound $\mathcal{H}^{m-1}(\partial X \cap B_1(p)) \leq C(m, \kappa)$. See also [BNS22] for implicit bounds in the setting of spaces with Ricci curvature bounded below under a further quantitatively noncollapsed assumption.

More generally we prove the following bound which is also sharp.

Theorem 1.7. Let X^m be Alexandrov of curv $\geq \kappa$. Let $R > 0$ and assume in addition that $R < \pi_\kappa/2$ if $\kappa > 0$. If $\overline{B_R(p)} = X$ for some $p \in X$ then

$$(1.8) \qquad\qquad \mathcal{H}^{m-1}(\partial X) \leq \mathcal{H}^{m-1}(\partial B_R^{\kappa}(o_p)),$$

where $B_R^{\kappa}(o_p)$ is the ball of radius R in T_pX equipped with the κ-cone metric (see section 2 for the definition).

The upper bound in Theorem 1.1 holds locally too.

Theorem 1.9. Let X^m be Alexandrov with curv $\geq \kappa$. Let $R > 0$ and assume in addition that $R \leq \pi_\kappa/2$ if $\kappa > 0$. Then for any $p \in X$ it holds that

$$\text{(1.10)} \qquad \mathcal{H}^{m-1}(\partial X \cap B_R(p)) \leq \mathcal{H}^{m-1}(\partial B_R^\kappa),$$

where B_R^κ is the ball of radius R in $M^m(\kappa)$.
  If $p \in \partial X$ then

$$\text{(1.11)} \qquad \mathcal{H}^{m-1}(\partial X \cap B_R(p)) \leq \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa),$$

where $B_{R,+}^\kappa$ is the half-ball of radius R in $M^m(\kappa)$. Moreover, in either of these cases the inequalities are strict unless $X = \overline{B_R(p)}$ and X is one of the spaces given by Theorem 1.12.

We also completely analyze the equality case in the setting of Theorem 1.1. For $\kappa > 0$, we will refer to the intersection of two closed hemispheres intersecting at an angle of α in $M^m(\kappa)$ as an Alexandrov lens and denote it by $L_\alpha^{m,\kappa}$ after the convention of [GP22]. Note that $L_\pi^{m,\kappa}$ is just the closed hemisphere in $M^m(\kappa)$.

Theorem 1.12. Let X^m be Alexandrov of curv $\geq \kappa$. Let $R > 0$ and assume that $\overline{B_R(p)} = X$ for some $p \in X$. If X satisfies

$$\text{(1.13)} \qquad \mathcal{H}^{m-1}(\partial X) = \mathcal{H}^{m-1}(\partial B_R^\kappa)$$

then $X = \overline{B_R^\kappa} \subset M^m(\kappa)$ if either $\kappa \leq 0$ or if $\kappa > 0$ and $R < \pi_\kappa/2$. In the case that $\kappa > 0$ and $R = \pi_\kappa/2$, X must be isometric to $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi]$.
  Furthermore, if $p \in \partial X$ and X satisfies

$$\text{(1.14)} \qquad \mathcal{H}^{m-1}(\partial X) = \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa),$$

then $X = \overline{B_{R,+}^\kappa} \subset M^m(\kappa)$ if either $\kappa \leq 0$ or if $\kappa > 0$ and $R < \pi_\kappa/2$. In the case that $\kappa > 0$ and $R = \pi_\kappa/2$, X must be $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi/2]$.

Rigidity in the case $\kappa = 1, R = \pi/2$ was first addressed and proved in [GP22]. However, their proof strongly uses the rigidity from the maximality of radius and does not generalize to arbitrary κ and R. Instead we use a much more analytical approach. We will use their rigidity result in one step of our argument but it should also be possible to use our methods to give an alternative proof for this case directly using induction. For the sake of brevity, we will not write out the details but will give a short discussion at the relevant section (see Remark 4.3).
  We also analyze almost rigidity.

Theorem 1.15. For any $\epsilon > 0$, there exists $\delta(\kappa, m, R)$ such that the following holds: Let X^m be Alexandrov with curv $\geq \kappa$. Let $R > 0$ and assume that $\overline{B_R(p)} = X$ for some $p \in X$. Assume X satisfies

$$\text{(1.16)} \qquad \mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B_R^\kappa) - \delta.$$

If $\kappa > 0$ and $R < \pi_\kappa/2$ or $\kappa \leq 0$ then

$$d_{GH}(X, \overline{B_R^\kappa}) < \epsilon.$$

If $\kappa > 0$ and $R = \pi_\kappa/2$, then there exists Y, which is either $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi]$ or the closed ball of radius R in $M^{m-1}(\kappa)$, so that

$$d_{GH}(X, Y) < \epsilon.$$

Furthermore, assuming that $p \in \partial X$ and that X satisfies

(1.17) $\mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa) - \delta.$

If $\kappa > 0$ and $R < \pi_\kappa/2$ or $\kappa \leq 0$ then

$$d_{GH}(X, \overline{B_{R,+}^\kappa}) < \epsilon.$$

If $\kappa > 0$ and $R = \pi_\kappa/2$, then there exists Y, which is either $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi/2]$ or the closed ball of radius R in $M^{m-1}(\kappa)$, so that

$$d_{GH}(X, Y) < \epsilon.$$

We comment briefly on the proof. In the case where $\kappa > 0$ and $R < \pi_\kappa/2$ or $\kappa \leq 0$, we rule out the possibility of collapse in the almost rigidity situation and so the previous theorem follows directly from standard Gromov-Hausdorff compactness along with rigidity. In the case where $\kappa > 0$ and $R = \pi_\kappa/2$, collapsing is clearly possible and so a more delicate analysis of collapsing limits is required to show that a collapse to the closed ball of radius $\pi_\kappa/2$ in $M^{m-1}(\kappa)$ is all that can happen.

1.1. **Open questions.** We end this section with some open questions. The most natural one is what can be said if lower sectional curvature bound is replaced by lower Ricci curvature bound. See [DPG18, KM21, BNS22] and references therein for the theory of the boundary of noncollapsed $\mathrm{RCD}(K, N)$ spaces.

Question 1.18. Let $(X, \mathrm{d}, \mathcal{H}^N)$ be a noncollapsed $\mathrm{RCD}(K, N)$ space. Suppose that $\mathrm{Rad}(X) = R$. Is there an upper bound on $\mathcal{H}^{m-1}(\partial X)$? If so is the sharp bound given by the volume of the boundary of an R ball in model space? Can one classify the rigidity cases?

The current best estimate of this type is given by [BNS22][Theorem 1.4], where they prove an implicit bound under a further quantitative noncollapsed assumption. More precisely, they prove the following.

Theorem 1.19. Let $(X, \mathrm{d}, \mathcal{H}^N)$ be a noncollapsed $\mathrm{RCD}(K, N)$ space. Let $p \in X$ so that $\mathcal{H}^N(B_1(p)) > v > 0$. Then for any $x \in \partial X \cap B_1(p)$ and $r \in (0, 1)$,

$$\mathcal{H}^{N-1}(B_r(x) \cap \partial X) \leq C(K, N, v)r^{N-1}.$$

However, it remains an open question whether the dependence on v can be removed and so even the existence of an implicit bound in Question 1.18 is not known.

In the case where X is Alexandrov with $\mathrm{curv} \geq \kappa > 0$ and with nonempty boundary, it is natural to assume that $\mathrm{Rad}(X) = R \leq \pi_\kappa/2$. This is because if $R > \pi_\kappa/2$ then $\partial X = \varnothing$ due to the Grove-Petersen radius sphere theorem [GP93]. For $\mathrm{RCD}(\kappa(N-1), N)$ spaces with $\kappa > 0$ we do not know if there is an example with $R > \pi_\kappa/2$ and $\partial X \neq \varnothing$.

Question 1.20. Let $(X, \mathrm{d}, \mathcal{H}^N)$ be a noncollapsed $\mathrm{RCD}(N-1, N)$ space with nonempty boundary. Is it possible that $\mathrm{Rad}(X) > \pi/2$?

A trivial but useful corollary of Theorem 1.9 (see the proof of Theorem 5.4) is that a uniform lower bound on the volume of the boundary along a convergent sequence of Alexandrov spaces bounds the dimension of collapse. More precisely, if we have a sequence of Alexandrov spaces $(X_i^m, \mathrm{d}_i, p_i)$ with curv $\geq \kappa$ converging to $(X_\infty, \mathrm{d}_\infty, p_\infty)$ and it holds that $\mathcal{H}^{m-1}(\partial X_i \cap B_1(p_i)) > c > 0$ for all i, then $\dim(X_\infty) \geq m - 1$. More generally, a uniform lower bound on $\mathcal{H}^k(\mathcal{S}_\epsilon^k(X_i) \cap B_1(p_i))$ would also bound the dimension of collapse using the implicit bounds of [LN20]. In the RCD case, a similar result would follow easily in the same way if the implicit bound of Theorem 1.19 can be given without dependence on v. Since this is not known, we have the following question.

Question 1.21. If $(X_i, \mathrm{d}_i, \mathcal{H}^N, p_i)$ are noncollapsed RCD(K, N) spaces converging to some $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, p_\infty)$ satisfying $\mathcal{H}^{N-1}(\partial X_i \cap B_1(p_i)) > c > 0$ for all i. Is it true that $\dim(X_\infty) \geq N - 1$?

This question can be asked for either $\dim(X_\infty)$ being the Hausdorff dimension or the essential dimension of X_∞. If a counterexample can be given for Question 1.21 for the Hausdorff dimension then it would immediately imply that the estimates of Theorem 1.19 cannot be made independent of v.

Finally, in the setting of Alexandrov spaces one can ask for optimal bounds on other geometric invariants of the boundary.

Question 1.22. Let (X^m, d) be Alexandrov with curv $\geq \kappa$ and $\mathrm{Rad}(X) = R$. What are the sharp bounds on the diameter and radius of a connected component of ∂X? What are the rigidity cases?

An implicit bound can be given using [Fuj18][Theorem 6.6], which more generally bounds the maximal number of ϵ-separated points on an extremal set with respect to its intrinsic geometry. Of course, sharp bounds would follow immediately in the case $\kappa = 1$ and $R = \pi/2$ from the conjecture that ∂X is Alexandrov. We mention that the diameter rigidity of the boundary for this case was analyzed in [GL21] under the additional assumption that ∂X is a smooth Riemannian manifold of dimension $m - 1$ with sec ≥ 1.

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2. Preliminaries

We begin by introducing some definitions and notations, see [BP92, AKP22] for a more careful treatment.

Given an Alexandrov space X with curvature bounded below by 1, we denote by $C_\kappa(X)$ the standard warped cone with curvature bounded below by κ. Notice that in the case of $\kappa > 0$ the cone is really a suspension where the radial parameter is between 0 and π_κ. We will usually write $C(X)$ for $C_0(X)$ since it is the standard Euclidean cone.

Fix now dimension $m \in \mathbb{N}$ and $\kappa > 0$. Let (X, d) be an Alexandrov space of dimension m and curvature lower bound κ. For each $p \in X$ we denote by $\Sigma_p X$ the space of directions at p and $(T_p X, o_p) = (C(\Sigma_p X), o_p)$ to be the tangent cone at p. When it is unambiguous we will often drop X in above notations and just write Σ_p and T_p.

Given $p, q \in X$, we will use $[pq]$ to denote a (usually not unique) geodesic between p and q. We will use $\log_p([pq]) \in T_p X$ to denote the vector corresponding to this geodesic, i.e., $\log_p([pq])$ is in the direction of $[pq]$ and has magnitude $\mathrm{d}(p, q)$.

On the closed ball $\overline{B_{\pi_\kappa/2}(o_p)} \subset T_p$, one may define a gradient exponentiation map $\mathrm{gexp}_p^\kappa : \overline{B_{\pi_\kappa/2}(o_p)} \to X$ which shares similar properties as the usual exponentiation map on $M^m(\kappa)$. The original construction is from [PP96]. We point out that the map depends on κ. For $v \in \Sigma_p$ the curve $\sigma(t) = \mathrm{gexp}_p^\kappa(tv)$ satisfies the IVP

$$(2.1) \qquad \begin{cases} \sigma'_+(t) = \frac{\tan_\kappa(\mathrm{d}(\sigma(t),p))}{\tan_\kappa(t)} \nabla_{\sigma(t)} \, \mathrm{d}(\cdot, p) \\ \sigma'_+(0) = v \end{cases}$$

where $\tan_\kappa := \mathrm{sn}_\kappa / \mathrm{cs}_\kappa$ is the generalized tangent function, see [AKP22][Section 1.A] for the definitions of sn_κ and cs_κ.

Further, $\sigma(t)$ is a reparameterized gradient curve of the modified distance function $f(\cdot) = \mathrm{itan}_\kappa(\mathrm{d}(\cdot, p))$, where $\mathrm{itan}_\kappa(t) := \int_0^t \tan_\kappa(s)ds$. The function f is double semiconcave (see Definition 6.11) on $B_R(p)$ by construction. We refer to the proof of [AKP22][16.31] for a careful discussion of these facts.

We outline some of the key properties of the gradient exponentiation map we shall need below and refer to [AKP22, Chapter 16] and [Pet07, Chapter 3] for the proofs. The first proposition says that that gexp_p^κ agrees with shortest geodesics where they exist.

Proposition 2.2. For any $q \in X$ and any geodesic $[pq]$ from p to q, $\mathrm{gexp}_p^\kappa(\log_p([pq])) = q$.

To state the next proposition we introduce the notation for comparison angles in $M^2(\kappa)$. For $a, b, c > 0$, we define $\tilde{\measuredangle}^\kappa(a; b, c)$ to be the angle corresponding to the triangle with side lengths a, b, and c in $M^2(\kappa)$ that is opposite of the side with length a. For obvious reason we require $a + b + c < 2\pi_\kappa$. In the case where the triangle inequality fails, for example if $b + a < c$ or if $c + a < b$, we will extend the usual definition so that $\tilde{\measuredangle}^\kappa(a; b, c) = 0$. Similarly, $\tilde{\measuredangle}^\kappa(a; b, c) = \pi$ if $b + c < a$. The following proposition says that the gradient exponentiation map satisfies similar comparison estimates as the exponentiation map on $M^m(\kappa)$ up until a radius of $\pi_\kappa/2$.

Proposition 2.3. Let $q_1, q_2 \in X$ so that $\mathrm{d}(p, q_i) \leq \pi_\kappa/2$ for each i and let $v_1, v_2 \in T_p$ be vectors which correspond to some geodesic from p to q_1 and q_2 respectively. Let $s_1, s_2 \geq 1$ so that $|s_i v_i| \leq \pi_\kappa/2$ for each i, then

$$\tilde{\measuredangle}^\kappa(\mathrm{d}(\mathrm{gexp}_p^\kappa(s_1 v), \mathrm{gexp}_p^\kappa(s_2 v)); |s_1 v_1|, |s_2 v_2|) \leq \tilde{\measuredangle}^\kappa(\mathrm{d}(q_1, q_2); |v_1|, |v_2|).$$

If we identify $B_{\pi_\kappa/2}(o_p) \subset T_p$ with $B_{\pi_\kappa/2}^\kappa(o_p) \subset C_\kappa(\Sigma_p)$ in the obvious way and use this to define an exponential map on the latter, which by abuse of notation we also denote gexp_p^κ, then from the previous proposition we immediately have the following.

Proposition 2.4. The map $\mathrm{gexp}_p^\kappa : B_{\pi_\kappa/2}^\kappa(o_p) \to X$ is 1-Lipschitz.

3. Upper bound on the volume of the boundary

In this section we prove Theorems 1.1, 1.7, and 1.9. We will focus on the first two theorems and give a brief discussion for the last one at the end of the section.

The main step is the following key Lemma which by Proposition 2.4 immediately implies the above theorems. In this lemma we only assume that X^m is Alexandrov with curv $\geq \kappa$ and $X = \overline{B_R(p)}$ for some $p \in X$.

Lemma 3.1. The boundary ∂X is contained in $\mathrm{gexp}_p^\kappa(\partial B_R(o_p))$.

Remark 3.2. The lemma generalizes the same statement for $\kappa = 1$ and $R = \pi/2$, which was used by Petrunin to solve Lytchak's problem in [Pet07][Proof of 3.3.5]. However, Petrunin's proof does not generalize to general κ and R and to the local version needed for Theorem 1.9. We give a different argument that does work in all of these cases.

Proof. First, recall that for any $v \in \Sigma_p$, the curve $t \mapsto \mathrm{gexp}_p^\kappa(tv)$ for $0 \leq t \leq R$ is a reparameterized gradient curve of the modified distance function $f(\cdot) = \mathrm{itan}_\kappa(d(\cdot, p))$, The function f is double semiconcave (see Definition 6.11) on $B_R(p)$ by construction.

We will consider two separate cases, depending on whether or not $p \in \partial X$.

Case 1 Suppose $p \notin \partial X$. Let $x \in \partial X$. We will show $x \in \mathrm{gexp}_p^\kappa(\partial B_R(o_p))$. Let $v \in \Sigma_p X$ be the initial direction of a shortest geodesic $[px]$.

If x is a critical point of $d(\cdot, p)$ (and hence of f), then by the definition of gexp_p^κ we have that $x = \mathrm{gexp}_p^\kappa(R \cdot v)$ and hence $x \in \mathrm{gexp}_p(S_R(o_p))$, where $S_R(o_p)$ is the sphere of radius R in $T_p X$. Notice that in this case $T_p X$ is without boundary so $S_R(o_p)$ is exactly $\partial B_R(o_p)$.

Suppose now that x is not a critical point of f and $d(\cdot, p)$. Note that this means that $d(p, x) < R$, though in any case what we would like to prove for x is obvious if $d(p, x) = R$.

Since ∂X is extremal, by [Fuj18] (cf. [Pet07][Section 2.2, Property 3]) there exists a gradient curve of f, which we denote $\gamma : (-\infty, t_{\max}) \to \partial X$, such that $\gamma(0) = x$. If $\kappa \leq 0$ then we may take $t_{\max} = \infty$. If $\kappa > 0$ and $t_{\max} < \infty$, then $\lim_{t \to t_{\max}} d(p, \gamma(t)) = \pi_\kappa/2$. In particular, $t_{\max} = \infty$ if $R < \pi_\kappa/2$. Note that there is a small $\varepsilon > 0$ such that $\gamma(t)$ is not a critical point of f for any $t \leq \varepsilon$ and so γ is a topological embedding of any finite interval contained in $(-\infty, \varepsilon]$.

Fix any $t < 0$ and let $q_t = \gamma(t)$. Let w_t be the initial direction of a shortest geodesic $[p, q_t]$ and let $d_t = d(p, q_t)$. Then $\mathrm{gexp}_p^\kappa(d_t \cdot w_t) = q_t$ and for any $d_t \leq s \leq \pi_\kappa/2$ we have that

$$\mathrm{gexp}_p^\kappa(s \cdot w_t) = \gamma(\tau(t, s))$$

for some continuous function $\tau(t, s)$ defined on $\{(t, s) : t < 0, d_t \leq s \leq \pi_\kappa/2\}$. Clearly, $\tau(t, d_t) = t$ and $\tau(t, s)$ is increasing in s.

We claim that there is a $t < 0$ such that $\tau(t, s) < 0$ for any $d_t \leq s \leq R$. Indeed we have that both $d(\cdot, p)$ and f are strictly increasing along $\gamma(t)$ for $-\infty < t < \varepsilon$. Since $p \notin \partial X$ and $\gamma \subset \partial X$ we have that $d(\gamma(t), p)$ is bounded below by a positive constant.

We have $f(\gamma(t))'_+ = |\nabla_{\gamma(t)} f|^2 = \tan_\kappa(d(\gamma(t), p))^2 |\nabla_{\gamma(t)} d(\cdot, p)|^2 \geq c |\nabla_{\gamma(t)} d(\cdot, p)|^2$. Therefore for sufficiently negative t we have that $\int_t^{t+R} |\nabla_{\gamma(s)} d(\cdot, p)|^2 ds$ can be made arbitrary small due to integrability reasons. By Cauchy-Schwarz the same is true for $\int_t^{t+R} |\nabla_{\gamma(s)} d(\cdot, p)| ds$. Now the speed of the radial curve $s \to \mathrm{gexp}_p^\kappa(sw_t)$ is bounded by $|\nabla d(\cdot, p)|$ at every point by (2.1), so we get that the length of this curve on the interval $[d_t, R]$ can be made arbitrary small for sufficiently negative t. By Lipschitz properties of f this implies that $f(\mathrm{gexp}_p^\kappa(s \cdot w_t)) < f(\gamma(0))$ for any $s \in [d_t, R]$ for sufficiently negative t. This means that by choosing sufficiently negative t we can ensure that $\tau(t, R) < 0$. We fix such a t_0.

Consider now the map $\phi\colon [t_0, 0] \to \mathbb{R}$ given by $\phi(t) = f(\gamma(\tau(t, R))) = f(\mathrm{gexp}_p^\kappa(R \cdot w_t))$. This map is continuous with $\phi(t_0) < f(\gamma(0))$ and $\phi(0) > f(\gamma(0))$. Hence by the Intermediate Value theorem there is a $t \in [t_0, 0]$ such that $\phi(t) = f(\gamma(0))$. Since f is strictly increasing along γ for $t \le 0$ it follows that $x = \gamma(0) = \mathrm{gexp}_p^\kappa(R \cdot w_t)) \in \mathrm{gexp}_p^\kappa(S_R(0))$, as claimed.

This finishes the proof in Case 1.

Case 2 Now suppose that $p \in \partial X$. We have that $\partial B_R^\kappa(o_p)$ consists of two parts: the sphere $S_R(o_p)$ and the closed ball of radius R in $T_p \partial X$ which is the κ-cone over $\partial \Sigma_p = \Sigma_p \partial X$. Let us denote this ball by $\bar{B}_R^\partial(o_p)$.

We need to show that $\partial X \subset \mathrm{gexp}_p^\kappa(\bar{B}_R^\partial(o_p) \cup S_R(o_p))$.

Suppose $x \in \partial X$ but $x \notin \mathrm{gexp}_p^\kappa(\bar{B}_R^\partial(o_p))$. If $\mathrm{d}(x, p) = R$ the statement of the lemma is obvious so assume $d(p, x) < R \le \pi_\kappa / 2$.

As in the proof of Case 1 we first construct a gradient curve of f, denoted $\gamma\colon (-\infty, t_{\max}) \to \partial X$ such that $\gamma(0) = x$. Then f is strictly increasing on $(-\infty, \varepsilon)$ and none of the points on this curve are critical for f.

There are two possibilities to consider. If $\lim_{t \to -\infty} f(\gamma(t)) > 0$ then this curve stays a definite distance away from p and the same proof as in Case 1 works.

Now suppose that $\lim_{t \to -\infty} f(\gamma(t)) = 0$. This means that $\mathrm{d}(\gamma(t), p) \to 0$ as $t \to -\infty$.

As before for any $t < 0$ let $q_t = \gamma(t)$, w_t be the initial direction of a shortest geodesic $[p, q_t]$ and $d_t := \mathrm{d}(p, q_t)$. Then $\mathrm{gexp}_p^\kappa(d_t \cdot w_t) = q_t$ and for any $d_t \le s \le \pi_\kappa / 2$ we have that

$$\mathrm{gexp}_p^\kappa(s \cdot w_t) = \gamma(\tau(t, s))$$

for some continuous τ.

The vectors w_t subconverge to some $w \in \Sigma_p \partial X$ as $t \to -\infty$ and hence $\mathrm{gexp}_p^\kappa(s \cdot w_t) = \gamma(\tau(t, s)) \to \mathrm{gexp}_p^\kappa(s \cdot w)$ as $t \to -\infty$ for all $s > 0$. In particular it holds for $s \le R$.

But $\mathrm{gexp}_p^\kappa(s \cdot w) \in \mathrm{gexp}_p^\kappa(\bar{B}_R^\partial(o_p))$ for $s \le R$ and hence none of these points can be equal to x. Hence $\mathrm{gexp}_p^\kappa(R \cdot w) = \gamma(T)$ for some $T < 0$. Now the same Intermediate Value Theorem argument as in Case 1 applies and shows that $x \in \mathrm{gexp}_p^\kappa(S_R(o_p))$. This concludes the proof in Case 2.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since gexp_p^κ is 1-Lipschitz on $B_R^\kappa(o_p)$ by Proposition 2.4, it is nonincreasing on $\mathcal{H}^{m-1}$ measure. This along with the previous lemma immediately implies Theorem 1.7. Using the maximal volume bound for $(m-1)$-dimensional Alexandrov spaces with curv ≥ 1 with and without boundary, Theorem 1.7 implies Theorem 1.1.

Observe now that in the above proof $\mathrm{d}(\cdot, p)$ is nondecreasing along γ. Hence if we only assume that $\gamma(0) \in \partial X \cap B_R(p)$ (without assuming $X = B_R(p)$) then we still have $\gamma(t) \in \partial X \cap B_R(p)$ for any $t \le 0$. With that observation the same proof as in the lemma gives that $\partial X \cap B_R(p) \subset \mathrm{gexp}_p^\kappa(\partial B_R(o_p))$. Arguing now as in the previous paragraph, we obtain Theorem 1.9.

4. Rigidity

In this section we prove Theorem 1.12. Assume that X satisfies the conditions of the Theorem. Roughly, there are three key steps (of uneven length and difficulty):

(1) Prove that the map $\mathrm{gexp}_p^\kappa|_{\partial B_R^\kappa(o_p)}$ is an isometry between $\partial B_R^\kappa(o_p)$ and ∂X equipped with the intrinsic metric.
(2) Prove that X must be a convex subset of the model space $M^m(\kappa)$.
(3) Analyze the rigidity of convex subsets of the model space $M^m(\kappa)$ with radius R and maximal boundary volume.

Let us proceed with the proof. Suppose that $X = \overline{B_R(p)}$ and $\mathcal{H}^{m-1}(\partial X) = \mathcal{H}^{m-1}(\partial B_R^\kappa)$. We start with the following lemma.

Lemma 4.1. The point p is a regular and gexp_p^κ maps $\partial B_R^\kappa(o_p)$ exactly onto ∂X.

Proof. By volume rigidity for Alexandrov spaces of $curv \geq 1$ it holds that $\mathcal{H}^{m-1}(\Sigma_p) \leq \mathcal{H}^{m-1}(\mathbb{S}^{m-1})$ and the equality is only possible if $\Sigma_p \cong \mathbb{S}^{m-1}$. Hence the by Theorem 1.7 the point p must be regular.

Next, suppose there exists some $v \in \partial B_R^\kappa(o_p)$ so that $\mathrm{gexp}_p^\kappa(v) \notin \partial X$. Since gexp_p^κ is 1-Lipschitz and ∂X is closed, we can find some small neighborhood U of v in $\partial B_R^\kappa(o_p)$ so that $\mathrm{gexp}_p^\kappa(U) \cap \partial X = \varnothing$. Then we have that

$$\mathcal{H}^{m-1}(\partial B_R^\kappa(o_p)) > \mathcal{H}^{m-1}(\partial B_R^\kappa(o_p) \setminus U) \geq \mathcal{H}^{m-1}(\mathrm{gexp}_p^\kappa(\partial B_R^\kappa(o_p) \setminus U)) \geq \mathcal{H}^{m-1}(\partial X),$$

by Lemma 3.1. Since $\mathcal{H}^{m-1}(\partial B_R^\kappa(o_p))$ is at most $\mathcal{H}^{m-1}(\partial B_R^\kappa)$, this contradicts assumption (1.13). $\qquad\square$

Since $\mathrm{gexp}_p^\kappa|_{B_R^\kappa(o_p)} : B_R^\kappa(o_p) \to X$ is 1-Lipschitz by Proposition 2.4, the map $\mathrm{gexp}_p^\kappa|_{\partial B_R^\kappa(o_p)} : \partial B_R^\kappa(o_p) \to \partial X$ must also be 1-Lipschitz, where $\partial B_R^\kappa(o_p)$ and ∂X are both equipped with the intrinsic metrics.

Let $\mathcal{H}_{\partial X}^{m-1}$ and $\mathcal{H}_X^{m-1}$ be the $(m-1)$-Hausdorff measures on ∂X with respect to the intrinsic metric $d_{\partial X}$ and the extrinsic metric d_X respectively. We will often implicitly make use of the following result due to Fujioka who proved the same more generally for extremal subsets [Fuj22, Corollary 3.17].

Lemma 4.2. The intrinsic and extrinsic $(m-1)$-Hausdorff measures on ∂X are equal:
$$\mathcal{H}_{\partial X}^{m-1} = \mathcal{H}_X^{m-1}.$$

Henceforth we will not distinguish between intrinsic and extrinsic $\mathcal{H}^{m-1}$ on ∂X.

Arguing as in Lemma 4.1 and using (1.13), Lemma 4.2 implies that $\mathrm{gexp}_p^\kappa|_{\partial B_R^\kappa(o_p)} : (\partial B_R^\kappa(o_p), d_{\partial B_R^\kappa(o_p)}) \to (\partial X, d_{\partial_X})$ is $\mathcal{H}^{m-1}$-preserving (in the sense that the $\mathcal{H}^{m-1}$ measure of a set is equal to that of its image). As noted before, it is also 1-Lipschitz and onto. We will prove from these three facts that it must be an isometry. This type of result is known as Lipschitz-volume rigidity.

Remark 4.3. In the case where the domain and target are Alexandrov spaces without boundary such a Lipschitz-volume rigidity statement was proved in [Li15]. However, that result cannot be applied directly since we do not know that ∂X is Alexandrov. Our proof will be simpler than that of [Li15] since we know that the domain is actually a smooth manifold without boundary. In this proof we will use the following consequence of the rigidity result of [GP22]: Let $\kappa = 1$ and $R = \pi/2$. If $curv\, X^m \geq \kappa$ and $\mathcal{H}^{m-1}(\partial X) = \mathcal{H}^{m-1}(\mathbb{S}^{m-1})$ then ∂X is isometric to $\mathbb{S}^{m-1}$ with respect to the intrinsic metrics.

Alternatively, this follows by induction on the dimension of X. More precisely. Proposition 4.6 in dimension $m-1$ implies Lemma 4.4 in dimension m which implies Lemma 4.5 in

dimension m which implies Proposition 4.6 in dimension m. Thus the use of [GP22] for proving Proposition 4.6 can actually be avoided.

We first prove that every point $p \in \partial X$ is regular with respect to the intrinsic metric $\mathrm{d}_{\partial X}$.

Lemma 4.4. For every point $q \in \partial X$, $T_q \partial X = (\mathbb{R}^{m-1}, \mathrm{d}_{\mathbb{R}^{m-1}})$.

Proof. Fix some $q \in \partial X$ and choose $q' \in \partial B_R^\kappa(o_p)$ so that $\mathrm{gexp}_p^\kappa(q') = q$. Since gexp_p^κ is 1-Lipschitz, we have that $\mathrm{gexp}_p^\kappa(B_R^\kappa(o_p) \cap B_r(q')) \subset B_r(q)$ (here we are considering extrinsic metrics). Now using that the map is $\mathcal{H}^{m-1}$-preserving from $\partial B_R^\kappa(o_p)$ to ∂X (note that it is not $\mathcal{H}^{m-1}$-preserving on all of $B_R^\kappa(o_p)$), we have that for arbitrary $r > 0$,

$$\mathcal{H}^{m-1}(\partial X \cap B_r(q)) \geq \mathcal{H}^{m-1}(\mathrm{gexp}_p^\kappa(\partial B_R^\kappa(o_p) \cap B_r(q'))) = \mathcal{H}^{m-1}(\partial B_R^\kappa(o_p) \cap B_r(q')).$$

Using that q' is the boundary point of a smooth Riemannian manifold, as $r \to 0$ we have

$$\frac{1}{r^{m-1}}\mathcal{H}^{m-1}(\partial X \cap B_r(q)) \geq \frac{1}{r^{m-1}}\mathcal{H}^{m-1}(\partial B_R^\kappa(o_p) \cap B_r(q')) \to \omega_{m-1},$$

where ω_{m-1} is the volume of the ball of radius 1 in $\mathbb{R}^{m-1}$. Taking a limit to the tangent cone at q and use the stability of volume for the boundary, for example from [BNS22][Theorem 1.8] or [Fuj22][Theorem 1.3], we have that

$$\mathcal{H}^{m-1}(B_1(o_q) \cap \partial(T_q X)) \geq \omega_{m-1}.$$

This means that the space of directions Σ_q at q, which has curv ≥ 1 and is with boundary, has maximal boundary volume. By the rigidity results of [Pet07] and [GP93], we conclude that the inequality in the previous equation is an equality and that Σ_q must be either a $(m-1)$-dimensional hemisphere or the nontrivial intersection of two such. In both cases, it is easy to see that $T_q \partial X = \partial(C(\Sigma_q))$ must be $\mathbb{R}^{m-1}$ under the intrinsic metric. $\square$

Next, we show that $\mathrm{gexp}_p^\kappa|_{\partial B_R^\kappa(o_p)} : \partial B_R^\kappa(o_p) \to \partial X$, which for notational simplicity we will also denote by gexp_p^κ, is a homeomorphism.

Lemma 4.5. The map $\mathrm{gexp}_p^\kappa : \partial B_R^\kappa(o_p) \to \partial X$ is a homeomorphism.

Proof. Since gexp_p^κ is continuous and onto, and $\partial B_R^\kappa(o_p)$ is compact, it suffices to show that gexp_p^κ is one-to-one. Fix $q \in \partial X$ and suppose for the sake of contradiction that there exist different points $q_1, q_2 \in \partial B_R^\kappa(o_p)$ so that $\mathrm{gexp}_p^\kappa(q_1) = \mathrm{gexp}_p^\kappa(q_2) = q$. Using that that gexp_p^κ is $\mathcal{H}^{m-1}$-preserving, we have that

$$\frac{1}{r^{m-1}}\mathcal{H}^{m-1}(B_r(q)) \geq \frac{1}{r^{m-1}}\mathcal{H}^{m-1}(\mathrm{gexp}_p^\kappa(B_r(q_1) \cup B_r(q_2)))$$

$$= \frac{1}{r^{m-1}}\mathcal{H}^{m-1}(B_r(q_1) \cup B_r(q_2)) \to 2\omega_{m-1},$$

for sufficiently small $r \to 0$, where the balls considered are with intrinsic distance. This is a contradiction since the intrinsic ball of radius r around q in ∂X is contained in the intersection of ∂X and the extrinsic ball of radius r around q, and so the lefthand side can converge to no more than ω_{m-1} by the previous lemma. $\square$

Finally, we will prove that the map is an isometry. For clarity, in the following we will use $B_r(\cdot)$ to indicate extrinsic balls and $B_r^\partial(\cdot)$ to indicate intrinsic balls.

Proposition 4.6. The map $\mathrm{gexp}_p^\kappa : \partial B_R^\kappa(o_p) \to \partial X$ is an isometry with respect to the intrinsic metrics.

Proof. Fix $q \in \partial X$ and its preimage $q' \in \partial B_R^\kappa(o_p)$ under gexp_p^κ. Obviously, we have that as $r \to 0$,

$$(4.7) \qquad \frac{1}{r^{m-1}} \mathcal{H}^{m-1}(B_r^\partial(q')) \to \omega_{m-1}.$$

We claim the same holds for q, that is, as $r \to 0$,

$$(4.8) \qquad \frac{1}{r^{m-1}} \mathcal{H}^{m-1}(B_r^\partial(q)) \to \omega_{m-1}.$$

Indeed, we have that $(X, \partial X, \frac{1}{r} \mathrm{d}_X, q) \to (T_q X, T_q \partial X, \mathrm{d}_{T_q X}, o_q)$ as $r \to 0$, so in particular by [BNS22, Theorem 1.8] or [Fuj22, Theorem 1.3], $\mathcal{H}^{m-1}$ on the boundaries converge as well. By Theorem 6.10, the boundaries converge under intrinsic metrics, i.e., $(\partial X, \frac{1}{r} \mathrm{d}_{\partial X}, q) \to (T_q \partial X, \mathrm{d}_{T_q \partial X}, o_q)$ as $r \to 0$. Combining this with the fact that $T_q \partial X = \mathbb{R}^{m-1}$ (Lemma 4.4) and the invariance of $\mathcal{H}^{m-1}$ with respect to intrinsic and extrinsic metric (Lemma 4.2) gives that

$$\frac{1}{r^{m-1}} \mathcal{H}^{m-1}(B_r(q) \cap \partial X) \to \omega_{m-1}.$$

To prove (4.8) from this all we need is the following lemma which says that for any $\delta > 0$, the intrinsic ball of radius r contains the extrinsic ball of radius $r/(1 + \delta)$ for sufficiently small r.

Lemma 4.9. Let (X^n, d) be an Alexandrov space, $E \subset X$ be an extremal subset and $p \in E$ be a point. Let d_E be the intrinsic metric on E induced by the extrinsic metric d. Then

$$\lim_{q \to p, q \in E} \frac{\mathrm{d}_E(p, q)}{\mathrm{d}(p, q)} = 1$$

Proof of Lemma 4.9. Let $f = \mathrm{d}(\cdot, p)$. It is well known that $\lim_{x \to p} |\nabla_x f| = 1$. Let $\delta > 0$. By above there is $r > 0$ such that $|\nabla_x f| > 1 - \delta$ for any $x \in B_r(p) \setminus \{p\}$.

Let $q \in E \cap B_r(p) \setminus \{p\}$. By [Fuj18, Lemma 6.1] there exists a backward gradient curve γ of f passing through q which lies in E. The function f decreases with speed at least $(1 - \delta)^2$ along this curve. This implies that γ reaches p in finite time $\leq \frac{\mathrm{d}(p,q)}{(1-\delta)^2}$. Since γ is 1-Lipschitz this gives that $\mathrm{d}_E(p, q) \leq \frac{\mathrm{d}(p,q)}{(1-\delta)^2}$.

$\square$

Now for each $r > 0$ consider the 1-Lipschitz pointed maps $\Phi_r : (\partial B_R^\kappa(o_p), \frac{1}{r} \mathrm{d}_{\partial B_R^\kappa(o_p)}, q') \to (\partial X, \frac{1}{r} \mathrm{d}_{\partial X}, q)$ which is just a rescaling of gexp_p^κ. Observe that the unit ball around q' (with respect to $\frac{1}{r} \mathrm{d}_{\partial B_R^\kappa(o_p)}$) gets more and more dense in the unit ball around q (with respect to $\frac{1}{r} \mathrm{d}_{\partial X}$). This follows by volume considerations from (4.7), (4.8) and the observation that for any $s > 0$ there is a uniform lower bound on the intrinsic ball of radius s around any $x \in \partial X$ (since it contains the image of the intrinsic ball of radius s around $(\mathrm{gexp}_p^\kappa)^{-1}(x)$ and gexp_p^κ is $\mathcal{H}^{m-1}$-preserving).

Taking now a subconvergent limit of Φ_r, we obtain some 1-Lipschitz $\Phi : T_{q'} \partial B_R(o_p) \to T_q \partial X$. Moreover, for any $x \in T_{q'}$ and $r > 0$, Φ maps $B_r(x)$ onto $B_r(\Phi(x))$ by the result

of the previous paragraph. Since both domain and codomain are isometric to $\mathbb{R}^{m-1}$, this implies that Φ is an isometry.

Fix $\delta > 0$. We claim that $(\mathrm{gexp}_p^\kappa)^{-1}$ does not increase the length of curves by more than a factor of $(1 + \delta)$. This will obviously imply that $(\mathrm{gexp}_p^\kappa)^{-1}$ is 1-Lipschitz and hence an isometry.

By a standard partitioning argument, it suffices to prove the following claim: For all sufficiently small r (depending on q and δ), if $\mathrm{d}_{\partial X}(x, q) = r$ then $\mathrm{d}_{\partial B_R^\kappa(o_p)}((\mathrm{gexp}_p^\kappa)^{-1}(y), q') \leq (1 + \delta)r$.

Since the map $\mathrm{gexp}_p^\kappa : \partial B_R^\kappa(o_p) \to \partial X$ is a homeomorphism the image of the metric sphere (with respect to intrinsic metric), we have that $\partial B_{(1+\delta)r}^\partial(q')$ separates ∂X.

Consider any shortest $[qy]$ in ∂X with respect to $\mathrm{d}_{\partial X}$ of length r. Since the map Φ above is an isometry we get that if r is sufficiently small this curve cannot intersect $\mathrm{gexp}_p^\kappa(\partial B_{(1+\delta)r}^\partial(q'))$. Since the latter separates ∂X it follows that the entire shortest $[q, y]$ including the point y lies in $\mathrm{gexp}_p^\kappa(B_{(1+\delta)r}^\partial(q'))$. This proves the claim and hence finishes the proof of Proposition 4.6. $\qquad\square$

Now we can prove that X must be a convex subset of the model space $M^m(\kappa)$.

Proposition 4.10. *X is isometric to a convex subset of the model space $M^m(\kappa)$.*

Proof. Fix any $x, y \in X$. Let $d = \mathrm{d}(x, y)$ and fix a unit speed geodesic $\gamma : [0, d] \to X$ between x and y. For each $t \in [0, d]$, we choose a direction $v(t) \in \Sigma_p = \mathbb{S}^{m-1}$ so that $\gamma(t)$ may be reached from p by a geodesic with direction $v(t)$ and length $s(t)$. In general, $v(t)$ may not be unique or continuously dependent on t, but $s(t)$ is obviously 1-Lipschitz. However, in our case $v(t)$ must be unique. This immediately follows from Lemma 4.5 which implies that the map $\mathrm{gexp}_p^\kappa : \partial B_R^\kappa(o_p) \to \partial X$ is injective.

Since $\Sigma_p = \mathbb{S}^{m-1}$, we may identify T_p isometrically with the tangent space at some designated origin $p' \in M^m(\kappa)$. Taking the Riemannian exponentiation map at p', we define $x' = \exp_{p'}(v(0), s(0))$ and $y' = \exp_{p'}(v(d), s(d))$. Denoting $d' = \mathrm{d}_{M^m(\kappa)}(x', y')$, we will prove that $d = d'$ and that $\tilde{\gamma}(t) := \exp_{p'}(v(t), s(t))$ is necessarily the geodesic between x' and y'. We will prove this in the case that $s(t) \neq 0$ for any $t \in [0, d]$. As can be seen, the other case will follow easily from the rest of our argument by dividing the geodesic into two parts at p and using a limiting argument.

We first show that $v(t)$ is Lipschitz on $[0, d]$. First consider the curve $\gamma_R(t) := \mathrm{gexp}_p^\kappa((v(t), R))$ in ∂X, where we identify $(v(t), R)$ to a point in $T_p = C(\Sigma_p)$ in the obvious way. It is easy to see that this curve must be Lipschitz by using Proposition 2.3 to compare $\mathrm{d}_X(\mathrm{gexp}_p^\kappa((v(t_1), R)), \mathrm{gexp}_p^\kappa((v(t_2), R))$ and $\mathrm{d}_X(\mathrm{gexp}_p^\kappa((v(t_1), s(t_1)), \mathrm{gexp}_p^\kappa((v(t_2), s(t_2)))) = \mathrm{d}(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$, giving the bound

$$\mathrm{d}_X(\mathrm{gexp}_p^\kappa((v(t_1), R), \mathrm{gexp}_p^\kappa((v(t_2), R)) \leq C_{\kappa, s_0, R}|t_2 - t_1|,$$

where $s_0 = \min_{t \in [0,d]} s(t) > 0$ by assumption. Now using that gexp_p^κ is an intrinsic isometry between $\partial B_R^\kappa(o_p)$ to ∂X by Proposition 4.6, we have that $\tilde{\gamma}_R(t) := (v(t), R)$ seen as a curve in $\partial B_R^\kappa(o_p)$ must also be Lipschitz on $[0, d]$. This immediately implies that $v(t)$ is Lipschitz viewed as a curve in Σ_p.

We now show that $d' = d$ and that $\exp_{p'}(v(t), s(t))$ is the geodesic between x' and y'. WLOG we can assume that $x \notin [py]$, $y \notin [px]$ and $p \notin [xy]$ since in any of these cases the equality $d = d'$ is obvious.

Using Proposition 4.6 again, we see that the curves γ_R and $\tilde{\gamma}_R$ must be of the same length. Denoting by $L(\cdot)$ the length of a locally Lipschitz curve, we claim the following.

Claim $L(\tilde{\gamma}) \leq L(\gamma) = d$.

Let us prove the Claim. Recall that a metric speed of a locally Lipschitz curve $\eta(t)$ in a metric space Y is defined as $|\eta'(t)| = \lim_{\varepsilon \to 0} \frac{d(\eta(t), \eta(t+\varepsilon))}{|\varepsilon|}$. Note that this notation does not mean that any kind of tangent vector to η exists. We have that the curves $\gamma, \tilde{\gamma}, \gamma_R$ and $\tilde{\gamma}_R$ are all Lipschitz. Therefore they have metric speeds defined for almost all t and their lengths are given by the integrals of the metric speeds [BBI01, Theorem 2.7.6] (this is obvious for γ because it is a geodesic). We denote by I the full measure the full measure set of $t \in (0, d)$ for which all four curves $\gamma, \tilde{\gamma}, \gamma_R, \tilde{\gamma}_R$ have metric derivatives.

Next we will show that $|\tilde{\gamma}'(t)| \leq |\gamma'(t)|$ for all $t \in I$ which will obviously imply the Claim. Let us fix $t \in I$ and let $a = |\tilde{\gamma}'_R(t)|$.

By above we have that $d(\tilde{\gamma}_R(t + \varepsilon), \tilde{\gamma}_R(t)) = a|\varepsilon| \pm o(|\varepsilon|)$ and the same holds for γ_R. Therefore by the cosine law in $M^m(\kappa)$ we have that comparison angles $\tilde{\angle}^\kappa(d(\gamma_R(t), \gamma_R(t + \varepsilon)); R, R)$ and $\tilde{\angle}^\kappa(d(\tilde{\gamma}_R(t), \tilde{\gamma}_R(t + \varepsilon)); R, R)$ differ by at most $o(|\varepsilon|)$.

Note that $\tilde{\angle}^\kappa(d(\tilde{\gamma}_R(t), \tilde{\gamma}_R(t + \varepsilon)); R, R) = \tilde{\angle}^\kappa(d(\tilde{\gamma}(t), \tilde{\gamma}(t + \varepsilon)); s(t), s(t + \varepsilon))$. On the other hand, by Proposition 2.3 it holds that

$$\tilde{\angle}^\kappa(d(\gamma(t), \gamma(t + \varepsilon)); s(t), s(t + \varepsilon)) \geq \tilde{\angle}^\kappa(d(\gamma_R(t), \gamma_R(t + \varepsilon)); R, R).$$

Combining the above we get

$$\tilde{\angle}^\kappa(d(\gamma(t), \gamma(t + \varepsilon)); s(t), s(t + \varepsilon)) \geq \tilde{\angle}^\kappa(d(\tilde{\gamma}(t), \tilde{\gamma}(t + \varepsilon)); s(t), s(t + \varepsilon)) - o(|\varepsilon|)$$

and hence

$$(4.11) \qquad\qquad d(\gamma(t), \gamma(t + \varepsilon)) \geq d(\tilde{\gamma}(t), \tilde{\gamma}(t + \varepsilon)) - o(|\varepsilon|).$$

By the definition of metric speed $d(\gamma(t), \gamma(t + \varepsilon)) = |\gamma'(t)|\varepsilon + o(|\varepsilon|)$ and $d(\tilde{\gamma}(t), \tilde{\gamma}(t + \varepsilon)) = |\tilde{\gamma}'(t)|\varepsilon + o(|\varepsilon|)$. Plugging these into (4.11), dividing by $|\varepsilon|$ and taking the limit as $\varepsilon \to 0$ gives

$$|\gamma'(t)| \geq |\tilde{\gamma}'(t)|.$$

Integrating from 0 to d gives the Claim.

Now $\tilde{\gamma}$ is a curve between x' and y' and so it has length at least d', which means $d' \leq d$. However, standard comparison tells us that $d' \geq d$. This means that $d = d'$ and $\tilde{\gamma}$ must be the geodesic from x' and y' as required. This means that the map $x \mapsto \log_p([px])$ gives a distance preserving embedding of X into the $\pi_\kappa/2$ ball around o_p in $T_p X$ equipped with the constant curvature κ metric and moreover the image is convex. Thus X is isometric to a convex subset of a $\pi_\kappa/2$ ball in $M^m(\kappa)$. $\qquad\square$

To finish the proof we need the following Theorem.

Theorem 4.12. Let $R \leq \pi_\kappa/2$ and let $p \in M^m(\kappa)$. Let $K_1 \subset K_2 \subset \overline{B_R(p)}$ be nested closed convex domains. Then

$$(4.13) \qquad\qquad \mathcal{H}^{m-1}(\partial K_1) \leq \mathcal{H}^{m-1}(\partial K_2)$$

Moreover, this inequality is strict if $R < \pi_\kappa/2$ and $K_1 \neq K_2$, i.e.,

$$(4.14) \qquad \mathcal{H}^{m-1}(\partial K_1) < \mathcal{H}^{m-1}(\partial K_2)$$

Remark 4.15. We could not initially find a reference for $\kappa > 0$ and so we gave a proof. It was later pointed out to the authors by Federico Glaudo that Theorem 4.12 follows from [Ban81][Section 3, Theorem 1].

Remark 4.16. The strict inequality can fail if $\kappa > 0$ and $R = \pi_\kappa/2$ even if $K_1 \neq K_2$. An example is when K_1 is an intersection of two hemispheres in $\mathbb{S}^n$ and K_2 is one of these hemispheres.

Proof. Theorem 4.12 easily follows from the following version of Crofton's formula in high dimensions [San76, 14.70]: Given a closed convex domain $K \subset B_{\pi_\kappa/2}(p)$ in $M^m(\kappa)$ with piecewise smooth boundary, the volume of the boundary of K can be computed by the formula

$$(4.17) \qquad \mathcal{H}^{m-1}(\partial K) = C(m) \cdot \int_{G(m)} \#\{L \cap \partial K\} dL$$

where $G(m)$ is the set of lines in $M^m(\kappa)$ (or great circles if $\kappa > 0$), $L \in G(m)$ and dL is some naturally defined volume measure on $G(m)$. The authors are grateful to Semyon Alesker for providing the reference to this formula.

Since the convex domains $K_1 \subset K_2$ are nested it follows that $\#\{L \cap \partial K_1\} \leq \#\{L \cap \partial K_2\}$ which immediately gives the non-strict inequality in Theorem 4.12 assuming that K_1 and K_2 have piecewise smooth boundaries.

When $R < \pi_\kappa/2$ the intersections $L \cap K_1$ and $L \cap K_2$ are intervals if nonempty and it is not hard to see that there is an open set of lines which intersect K_2 but not K_1 which gives the strict inequality (4.14).

The case of general K_1, K_2 easily follows by approximation. $\qquad \square$

Remark 4.18. It is instructive to think about why the above argument doesn't give a strict inequality if $\kappa > 0$ and $R = \pi_\kappa/2$ even if $K_1 \neq K_2$. This should give an alternate proof to [GP22] of the rigidity classification in this case.

Remark 4.19. A different proof of Theorem 4.12 for $\kappa \leq 0$ follows from a classical result that in a $CAT(0)$ space the nearest point projection onto a closed convex subset is 1-Lipschitz. The case $\kappa > 0$ can be shown using the proof of Theorem 1.1 in [LP21] which implies that there exists a strong deformation retraction $\Pi_t\colon K_2 \to K_2, t \in [0,1]$ where $\Pi_0 = \mathrm{Id}, \Pi_1\colon K_2 \to K_1$ and $\Pi_t|_{K_1} = \mathrm{Id}$ for all t. Moreover, Π_t is 1-Lipschitz for all t. Moreover if $R < \pi_\kappa/2$ then the constructed map Π_1 has local Lipschitz constant strictly smaller than 1 outside K_1. This gives the strict bound (4.14).

We now complete the proof of the rigidity theorem.

Proof of the first part of Theorem 1.12. By Proposition 4.10, X is a convex subset in $M^m(\kappa)$. It is clearly contained in the ball $\overline{B_R^\kappa(p)}$ in $M^m(\kappa)$. If it is not equal to the whole ball and $R < \pi_\kappa/2$ then by Theorem 4.12 $\mathcal{H}^{m-1}(\partial X) < \mathcal{H}^{m-1}(\partial B_R^\kappa)$, which is a contradiction. This finishes the proof of Theorem 1.12 in that case.

If $\kappa > 0$ and $R = \pi_\kappa/2$ then the result follows from [GP22]. $\qquad \square$

The second part of Theorem 1.12, i.e., when $p \in \partial X$, $\overline{B_R(p)} = X$, and $\mathcal{H}^{m-1}(\partial X) = \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa)$, can also be proved in a similar manner with some minor changes. We will not go into full detail of the proof but instead give an outline indicating the necessary changes to the previous proof.

Suppose that the conditions of the theorem are met.

First, one can show that $T_p X = \mathbb{R}_+^m$ and that the map $\mathrm{gexp}|_{\partial B_R^\kappa(o_p)}$ maps exactly onto the ∂X. Indeed, arguing as in the proof of Lemma 4.1, Lemma 3.1 implies that $\mathcal{H}^{m-1}(\partial B^\kappa(o_p)) = \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa)$. It follows that the double (see Theorem 6.7) of $\Sigma_p X$ must have the same measure as $\mathbb{S}^{m-1}$ and hence by the maximal volume rigidity for Alexandrov spaces with curv ≥ 1 must be isometric to $\mathbb{S}^{m-1}$. Therefore, $\Sigma_p X$ is the hemisphere and $T_p X = \mathbb{R}_+^m$. The proof that $\mathrm{gexp}|_{\partial B_R^\kappa(o_p)}$ maps exactly onto ∂X is exactly the same.

Next, one can show that that map $\mathrm{gexp}_p^\kappa : \partial B_R^\kappa(o_p) \to \partial X$ is an isometry. The proofs of Lemma 4.5 and 4.6 goes through without change. The only place where one has to be careful is that $B_R^\kappa(o_p)$ is not a manifold with smooth boundary anymore. However, it is clear that for any singular q, i.e., if q is exactly on the boundary of the R-hemisphere of o_p, one still has that q is a regular point of $\partial B_R^\kappa(o_p)$ and hence $\mathcal{H}^{m-1}(B_r(q) \cap \partial X) \to \omega_{m-1}$ as $r \to 0$, which is all that is needed.

Finally, the same proof for Proposition 4.10 shows that X must be a convex subset of $B_{R,+}^\kappa \subset M^m(\kappa)$. Theorem 4.12 now gives the desired result in the cases where $\kappa \leq 0$ or $\kappa > 0$ and $R < \pi_\kappa/2$. In the case where $\kappa > 0$ and $R = \pi_\kappa/2$, one has $\mathcal{H}^{m-1}(\partial B_{R,+}^\kappa) = \mathcal{H}^{m-1}(\partial B_R^\kappa)$, so one is actually in the original rigidity situation addressed in [GP22]. It is then easy to see that if $p \in \partial X$ and $X = \overline{B_{\pi_\kappa/2}(p)}$ then X must be $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi/2]$.

5. Almost Rigidity

Given the rigidity result we obtain in Theorem 1.12 it is natural to wonder what happens in the almost rigidity situation. We will consider the first part of Theorem 1.15 first. We will divide our analysis into two cases, when $R < \pi_\kappa/2$ and when $R = \pi_\kappa/2$.

We consider the case $R < \pi_\kappa/2$ first and prove the following.

Theorem 5.1. Given an integer m, $\kappa \in \mathbb{R}$ and $R > 0$. If $\kappa > 0$, assume in addition that $R < \pi_\kappa/2$. For any $\varepsilon > 0$, there exists $\delta(\kappa, m, R) > 0$ such that the following holds: Let X^m be Alexandrov with curv $\geq \kappa$. Assume $\overline{B_R(p)} = X$ for some $p \in X$ and that

$$\mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B_R^\kappa) - \delta.$$

Then

$$\mathrm{d}_{GH}(X, \overline{B_R^\kappa}) < \varepsilon.$$

Remark 5.2. Since B_R^κ is a topological disk for any $\kappa \in R$ and $R < \pi_\kappa/2$, by Perelman's Stability Theorem [Per91, Kap07] it follows that in the setting of the above theorem X must be a topological disk too provided δ is sufficiently small.

Proof. Suppose the theorem is false. Then there exist $\epsilon > 0$ and a contradicting sequence (X_i^m, d_i) of Alexandrov spaces with curv $\geq \kappa$ and $p_i \in X_i$ such that $B_R(p_i) = X_i$, $\mathcal{H}^{m-1}(\partial X_i) > \mathcal{H}^{m-1}(\partial B_R^\kappa) - \delta_i$ where $\delta_i \to 0$ as $i \to \infty$ but $\mathrm{d}_{GH}(X_i, \overline{B_R^\kappa}) \geq \varepsilon$ for all i.

By compactness, passing to a subsequence we may assume that $(X_i, \mathrm{d}_i, p_i) \to (X_\infty, \mathrm{d}_\infty, p_\infty)$ in Gromov-Hausdorff topology for some X_∞. We claim the following:

Claim $\dim X_\infty = m$, i.e., this sequence is noncollapsing.

Assuming the claim is true we see that X_∞ also has curv $\geq \kappa$ and satisfies $B_R(p_\infty) = X_\infty$. From the stability of $\mathcal{H}^{m-1}$ from [BNS22][Theorem 1.8] or [Fuj22][Theorem 1.1] we have that $\mathcal{H}^{m-1}(\partial X_i) \to \mathcal{H}^{m-1}(\partial X_\infty)$ and hence by Theorem 1.1 we have that $\mathcal{H}^{m-1}(\partial X_\infty) = \mathcal{H}^{m-1}(\partial B_R^\kappa)$. Now by Theorem 1.12, it follows that X is isometric to $\overline{B_R^\kappa}$. This is a contradiction with the assumption $d_{GH}(X_i, \overline{B_R^\kappa}) \geq \varepsilon$ for all i.

Thus to finish the proof of the theorem we need only to establish the Claim above that the sequence does not collapse.

Suppose that it is false and $\dim X_\infty = k < m$.

For each i let $d_i = d_i(p_i, \partial X_i)$. There are two possibilities:

Case 1 $\liminf_{i\to\infty} d_i = 0$ or

Case 2 $d_i \geq d > 0$ for all i for some $d > 0$.

Let us treat **Case 1** first.

By passing to a subsequence we can assume $d_i \to 0$. Then there exists $q_i \in \partial X_i$ such that $d(p_i, q_i) \leq d_i$. Then $B_{R+d_i}(q_i) = X_i$, and so by Theorem 1.1 we have that $\mathcal{H}^{m-1}(\partial X_i) \leq \mathcal{H}^{m-1}(\partial B_{R+d_i,+}^\kappa)$.

But $\mathcal{H}^{m-1}(\partial B_{R+d_i,+}^\kappa) \to \mathcal{H}^{m-1}(\partial B_{R,+}^\kappa) < \mathcal{H}^{m-1}(\partial B_R^\kappa)$. This contradicts the assumption that $\mathcal{H}^{m-1}(\partial X_i) > \mathcal{H}^{m-1}(\partial B_R^\kappa) - \delta_i$ where $\delta_i \to 0$. Therefore, Case 1 cannot occur.

We remark here that this argument fails if $\kappa > 0, R = \pi_\kappa/2$ since in this case $\mathcal{H}^{m-1}(\partial B_{R,+}^\kappa) = \mathcal{H}^{m-1}(\partial B_R^\kappa)$.

We now consider **Case 2**. Clearly we will have shown that Case 2 is impossible as soon as we prove the following.

Claim $\partial X_i \to X_\infty$ with respect to the ambient metrics on ∂X_i.

To prove the claim we will need the following lemma.

Lemma 5.3. Let Y^n be an Alexandrov space and $E \subset Y$ be extremal. If $\dim E = n$ then $E = Y$.

Here by $\dim E$ we mean either the topological or the Hausdorff dimension of E. They are the same by [PP93].

Proof. If $\dim E = n$ then by [Fuj22, Theorem 1.1] there is a regular point $p \in Y$ such that a small ball around p is contained in E. Let $y \in Y$ be arbitrary. Consider a shortest geodesic $[py]$. Then there is a point x on this geodesic close to p which is in E. let $f = d^2(\cdot, p)$. It is double semiconcave and since E is extremal it is invariant under the gradient flow ϕ_t of f. Hence $\phi_t(x) \in E$ for any $t \geq 0$. Since $x \in [py]$ there exists $t \geq 0$ such that $\phi_t(x) = y$. hence $y \in E$. □

Let us proceed with the proof of the **Claim** above.

After passing to a subsequence ∂X_i subconverges to an extremal subset E of X_∞, see, for example, [Pet07][Lemma 4.1.3]. If this is a proper subset then by Lemma 5.3 $\dim E < \dim X_\infty < m$ which means that $\dim E \leq m - 2$. Theorem 1.9 means that there is a universal $c = c(m) > 0$ such that for all $x \in X_i, r \leq \pi$ it holds that $\mathcal{H}^{m-1}(\partial X_i \cap B_r(x)) \leq c(m)r^{m-1}$. Now a simple covering argument implies that $\mathcal{H}^{m-1}(\partial X_i) \to 0$. Indeed, E has finite $\mathcal{H}^{m-2}$ by [Fuj18, Theorem 1.1]. Hence there is a $C > 0$ such that for any $\varepsilon > 0$, E can be covered by countably many balls $B_{r_j}(y_j)$ with $r_i < \varepsilon$ such that $\sum_j r_j^{m-2} < C$. Since E is compact

this collection of balls can be made finite so that $E \subset \cup_{j=1}^{N} B_{r_j}(y_j)$. Note that there exists a small $\mu > 0$ such that $\cup_{j=1}^{N} B_{r_j}(y_j)$ contains the μ-neighborhood of E.

These balls can be lifted to nearby balls $B_{r_j}(y_j^i)$ in X_i which will cover ∂X_i for all large i. Therefore,

$$\mathcal{H}^{m-1}(\partial X_i) \leq \sum_{j=1}^{N} \mathcal{H}^{m-1}(\partial X_i \cap B_{r_j}(y_j^i)) \leq \sum_{j=1}^{N} c(m) r_j^{m-1} \leq \sum_{j=1}^{N} c(m) \varepsilon r_j^{m-2} < \varepsilon c(m) C$$

for all large i. This shows that $\mathcal{H}^{m-1}(\partial X_i) \to 0$ if $E \neq X_\infty$, which is a contradiction. Therefore, Case 2 is impossible.

This proves the Claim that the convergence $X_i \to X$ is noncollapsing which finishes the proof of Theorem 5.1. □

We now consider almost rigidity in the case $\kappa > 0, R = \pi_\kappa/2$. We will prove

Theorem 5.4. Given an integer m and some $\kappa > 0$. Let $R = \pi_\kappa/2$. For any $\varepsilon > 0$, there exists $\delta(\kappa, m) > 0$ such that the following holds: Let X^m be Alexandrov with curv $\geq \kappa$. Assume $\overline{B_R(p)} = X$ for some $p \in X$ and that

$$\mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B_R^\kappa) - \delta.$$

Then there exists Y, which is either $L_\alpha^{m,\kappa}$ for some $\alpha \in (0, \pi]$ or the closed ball of radius R in $M^{m-1}(\kappa)$, so that

$$\mathrm{d}_{GH}(X, Y) < \varepsilon.$$

Proof. To unburden the exposition we skip some technical details in the proof.

By rescaling we need only consider the case $\kappa = 1$. Suppose we have $\mathcal{H}^{m-1}(\partial X_i) \to \mathcal{H}^{m-1}(\mathbb{S}^{m-1})$. If $\dim X_\infty = m$ then Theorem 1.12 gives that X_∞ is $L_\alpha^{m,1}$ for some $\alpha \in (0, \pi]$.

Suppose then that $\dim X_\infty = k < m$. The local bound given by Theorem 1.9 means that there is a universal $c = c(m) > 0$ such that for all $x \in X_i, r \leq \pi$ it holds that $\mathcal{H}^{m-1}(\partial X_i \cap B_r(x)) \leq cr^{m-1}$. This easily implies that if $k < m-1$ then $\mathcal{H}^{m-1}(\partial X_i) \to 0$ (see for example the computation in the previous proof). This gives a contradiction and hence $k < m-1$ is impossible. Thus $k = m-1$ is the only possibility if collapse does occur.

By the claim above $\partial X_i \to X_\infty$ with respect to ambient metrics, i.e., ∂X_i becomes denser and denser in X_i.

Let $q_\infty \in X_\infty$ be a regular point. Fix a small $\delta > 0$. Pick $q_i \in X_i$ converging to q_∞. Then by looking at an ϵ-strainer map $F \colon X_\infty \to \mathbb{R}^{m-1}$ near q_∞ and lifting it to $F_i \colon X_i \to \mathbb{R}^{m-1}$ we get that for all large i near q_i, F_i is a bundle map with fiber a compact connected 1-dimensional MCS-space. This means it's a finite graph.

By Corollary 6.18 it cannot contain any vertices of degree ≥ 3 and hence the fiber must be an interval. The boundary points of the fiber intervals are exactly the points in ∂X_i. Thus, restricted to ∂X_i near q_i we get that F_i is a 2-fold product cover of a disk in $\mathbb{R}^{m-1}$ and on each connected component it is a $(1 + \delta)$ bilipschitz homeomorphism [Fuj22, Proposition 3.3].

The set $S = X_\infty \setminus X_\infty(m-1, \delta)$, where $X_\infty(m-1, \delta)$ denote the points which are (m-1) δ-strained, is compact and has Hausdorff dimension $\leq m-2$. Take a small $\varepsilon > 0$. By the same argument as in the proof of Case 2 earlier we can find finitely many open balls $B_{r_j}(y_j), j = 1, \ldots, N$, covering S such $\sum_{j=1}^{N} r_j^{m-1} < \varepsilon$ and hence the corresponding balls

$B_{r_j}(y^i_j)$ have the property that $\mathcal{H}^{m-1}(\partial X_i \cap (\cup_j B_{r_j}(y^i_j))) \leq c(m)\varepsilon$. Outside of the union of these balls by a standard argument [BP92, Yam96] the maps $F^{-1} \circ F_i$ can be glued into a global map $\Phi_i \colon X_i \to X_\infty$ which is locally $(1 + \delta)$-bilipschitz from $\partial X_i \setminus \cup_j B_{r_j}(y^i_j)$ to $X_\infty \setminus \cup_j B_{r_j}(y_j)$. By above it is also 2-to-1. Since ε and δ can be chosen to be arbitrarily small this gives that $\mathcal{H}^{m-1}(X_\infty) = \frac{1}{2}\mathcal{H}^{m-1}(\mathbb{S}^{m-1})$.

Since curv $X_\infty \geq 1$ and $\overline{B_{\pi_\kappa/2}(p_\infty)} = X_\infty$, by absolute volume comparison we have that $\mathcal{H}^{m-1}(B(p_\infty, \pi/2)) \leq \frac{1}{2}\mathcal{H}^{m-1}(\mathbb{S}^{m-1})$. Thus we have an equality in the absolute volume comparison. This gives [Li15, Theorem 5.2] that X_∞ must be either isometric to a hemisphere in $\mathbb{S}^{m-1}$ or to a quotient of such hemisphere by an isometric involution of its boundary. This can also be thought of the quotient of $\mathbb{S}^{m-1}$ by an isometric involution of $\mathbb{R}^m$ which is given by a diagonal matrix with ± 1 on the diagonal.

Such quotients are easily understood depending on the number of -1's on the diagonal. There must be at least one -1. If it's exactly one we get a hemisphere $\mathbb{S}^{m-1}_+$. If all are -1's we get $\mathbf{RP}^{m-1}$. In between we get spherical joins of $\mathbf{RP}^k$ and $\mathbb{S}^l$ where $k + l = m - 2$.

A hemisphere can occur as the limit of Grove-Petersen examples. The case $X_\infty = \mathbf{RP}^{m-1}$ can be ruled out because it's not simply connected and X_∞ must be simply connected since all X_i are (see Lemma 5.5 below). It remains to rule out the intermediate cases.

It is well known that any compact Alexandrov space Y admits a nondecreasing contractibility function $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(r) \geq r$ and $\rho(r) \to 0$ as $r \to 0$. This means that there is $r_0 > 0$ such that for all any $r < r_0$ every ball $B_r(y)$ in Y is contractible inside the ball $B_{\rho(r)}(y)$. Moreover by [Pet07], [Kap02] or [MY14] there is a stronger contractibility statement: every ball $B_r(y)$ is contained in a closed set $C_r(y)$ of diameter $\leq \rho(r)$ where $C_r(y)$ is a superlevel set of a strongly concave function. In particular all sets $C_r(y)$ and their intersections are strongly convex and contractible. In particular Y is homotopy equivalent to the nerve of any finite covering by $C_r(y)$'s. For a stronger statement see [MY19].

Also, using the contractibility function it follows that any two sufficiently close maps from a finite dimensional CW complex to Y are homotopic (the required closeness might a priori depend on the dimension of the CW complex.)

This easily implies that given a convergent sequence of Alexandrov spaces $X^m_i \to X_\infty$ with curv $\geq \kappa$, diam $\leq D$ there are well defined maps $\pi_k(X_i) \to \pi_k(X_\infty)$ for large i [Per97]. These are obtained by using a fine triangulation of a spheroid in X_i and using contractibility radius to fill in the corresponding triangulation to get a spheroid in X_i, and the same with the homotopy of spheroids.

In general the map $\pi_k(X_i) \to \pi_k(X_\infty)$ need not be surjective as not every spheroid in X_∞ can be lifted to X_i. One obvious exception is when $k = 1$ since any loop in X_∞ can be lifted to a loop in X_i using a piecewise approximation by broken geodesics.

This gives

Lemma 5.5. Let $X^m_i \to X$ be a convergent sequence of Alexandrov spaces with with curv $\geq \kappa$, diam $\leq D$. Then for all large i there is an epimorphism $\pi_1(X_i) \to \pi_1(X_\infty)$. In particular if all X_i are simply connected then so is X_∞.

Note that the lemma is easily seen to fail for pointed convergence to a noncompact limit.

Remark 5.6. In general Lemma 5.5 works not just for Alexandrov spaces but in a much broader setting where one has a π_1 contractibility function for X, i.e. X has the property

that any loop contained in a ball of radius r is contractible in a ball of radius $\rho(r)$ where $\rho(r) \to 0$ as $r \to 0$. In particular it applies to Ricci limits [Wan21, Theorem 1.2].

Lemma 5.5 in particular applies in the case we are interested in when all X_i have curv ≥ 1 and nonempty boundary. Then all X_i are contractible and in particular simply connected. hence X_∞ is simply connected too. This rules out the possibility that $X_\infty = \mathbf{RP}^{m-1}$.

Now consider $X_\infty = \mathbf{RP}^k * \mathbb{S}^l$ with $k + l = m - 2$ and $k > 0, l \geq 0$.

We claim that in this case every spheroid in X_∞ can be lifted to X_i for all large i.

Let $f \colon \mathbb{S}^n \to X_\infty$ be a spheroid. We can choose a very fine triangulation of f and try to lift simplices skeleta by skeleta. As was explained it's easy to construct a lifting on the 1-skeleton. To do an induction step we need to be able to fill a simplex $f_i \colon \Delta^s \to X_i$ provided a map on its boundary has already been constructed. This is easy to do near regular points in X_∞ since the regular fiber is an interval and hence small balls in X_i close to the balls around regular points in X_∞ are contractible. However, this argument fails near singular points of X_∞ and a different argument is required there. The singular set in X is an isometric copy of $\mathbb{S}^l$ which is extremal with normal spaces of directions isometric to $\mathbf{RP}^k$.

Let us assume for simplicity that $l = 0$. Then X_∞ is the spherical suspension $S(\mathbf{RP}^{m-2})$ over $\mathbf{RP}^{m-2}$ and there are only two isolated singular points - the vertices of the suspension. The general case is similar to this one but locally everything is crossed with a disk in $\mathbb{R}^l$.

Let p be one of the vertices of $X_\infty = S(\mathbf{RP}^{m-2})$ and let $p_i \in X_i$ converge to p.

Claim: There is a small $r_0 > 0$ such that for any $r < r_0$ the points $p_i \to p$ can be chosen so that balls $B_r(p_i)$ are contractible for all large i.

This obviously allows filling in simplexes in $B_r(p_i)$ and hence shows that f can be lifted to a nearby spheroid $f_i \colon S^n \to X_i$. Hence for all large i the map $\pi_n(X_i) \to \pi_n(X_\infty)$ is onto. This leads to a contradiction since X_i is contractible and $S(\mathbf{RP}^{m-2})$ is not. More specifically $\pi_2(S(\mathbf{RP}^{m-2})) \cong H_2(S(\mathbf{RP}^{m-2})) \cong \mathbb{Z}_2 \neq 0$.

Thus it remains to prove the Claim above.

We can choose r_0 small enough so that for all $r < r_0$ the unit ball around the base point in $(X_\infty, \frac{1}{r} d_\infty, p)$ is $o(1)$-close to $B_1(o_p)$ in $T_p X_\infty \cong C(\mathbf{RP}^{m-2})$. By the work of Yamaguchi-Shioya [SY00] and Yamaguchi [Yam12] collapsing near p can be understood as follows.

Theorem 5.7. Let $(X_i^m, \mathrm{d}_i, q_i) \to (X_\infty, \mathrm{d}_\infty, q_\infty)$ where all X_i have curv $\geq \kappa$. Let $p \in X_\infty$. Then for all sufficiently small r there is $p_i \in X_i$ converging to p such that one of the following two possibilities holds:

(1) After passing to a subsequence $\mathrm{d}_i(\cdot, p_i)$ has no critical points in $B_r(p_i) \setminus \{p_i\}$ or
(2) There is a sequence $\delta_i \to 0$ such that
    (a) For any $\lambda > 1$ for all sufficiently large i the function $d(\cdot, p_i)$ has no critical points in the annulus $\{\lambda \delta_i \leq d(x, p_i) \leq r\}$
    (b) for any subsequential limit $(X_i, \frac{1}{\delta_i} \mathrm{d}_i, p_i) \to (Y, \mathrm{d}_Y, y_0)$ it holds that $\dim Y \geq \dim X_\infty + 1$.

Note that we obviously have that in the above theorem Y has curv ≥ 0.

We want to apply this theorem in our situation. If the first alternative in Theorem 5.7 holds then as discussed we can lift any spheroid from X_∞ to X_i which eventually leads to a contradiction. Let us examine what happens if the second alternative holds. Since in our case $\dim X_\infty = m - 1$ we must have that $\dim Y = m$. This means that the convergence $(X_i, \frac{1}{\delta_i} \mathrm{d}_i, p_i) \to (Y, \mathrm{d}_Y, y_0)$ is noncollapsing.

It is easy to see that the ideal cone $Y(\infty)$ of Y given by the limit $(Y, \frac{1}{\delta_i} \mathrm{d}_Y, y_0) \to (Y(\infty), \mathrm{d}_{Y(\infty)}, o_y)$ as $\delta_i \to 0$ has dimension at least $m-1$. This implies that the soul of Y has dimension 1 or 0. If it has dimension 0 then Y is contractible which by stability implies that all balls $\bar{B}_r(p_i)$ are contractible for large i and we are again in the situation where we can lift spheroids.

Let us examine what happens if the soul S of Y is 1-dimensional. Then S is $\mathbb{S}^1$ and by a standard argument using the splitting theorem we get that Y is a flat bundle over S_1. More precisely, there is a nonnegatively curved Alexandrov space Z^{m-1} and a point $z_o \in Z$ (the soul of Z) such that Y is isometric to $([0,1] \times Z)/\sim$ where we glue $\{0\} \times Z$ to $\{1\} \times Z$ by some isometry $\phi \colon Z \to Z$ which fixes z_0. This in particular implies that for all large R the spheres $S_R(y_0)$ have infinite fundamental groups. By the stability theorem and Morse theory this implies that for all large i the spheres $S_r(p_i)$ have infinite fundamental groups as well. But on the other hand the structure of collapsing away from p can be understood by the fibration theorem since M_∞ is smooth there. This gives that metric spheres $S_r(p_i)$ are homotopy equivalent to the total space of an interval bundle over $S_r(p)$ which is homeomorphic to $\mathbf{RP}^{m-1}$. This means that $\pi_1(S_r(p_i))$ must be finite for all large i. This is a contradiction which shows that this case is impossible.

This means that the case $X_\infty = \mathbf{RP}^k * \mathbb{S}^l$ with $k + l = m - 2$ and $k > 0, l \geq 0$ cannot occur at all and the only possibility for X_∞ is the round hemisphere $\mathbb{S}^{m-1}_+$. $\square$

Next we consider the second part of Theorem 1.15. The case where $\kappa > 0$ and $R = \pi_\kappa/2$ follows easily from the first part of the Theorem since $\mathcal{H}^{m-1}(\partial B^\kappa_{R,+}) = \mathcal{H}^{m-1}(\partial B^\kappa_R)$. We have

Theorem 5.8. Given an integer m and some $\kappa > 0$. Let $R = \pi_\kappa/2$. For any $\varepsilon > 0$, there exists $\delta(\kappa, m) > 0$ such that the following holds: Let X^m be Alexandrov with curv $\geq \kappa$. Assume that $\overline{B_R(p)} = X$ for some $p \in \partial X$ and that

$$\mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B^\kappa_{R,+}) - \delta.$$

Then there exists Y, which is either $L^{m,\kappa}_\alpha$ for some $\alpha \in (0, \pi/2]$ or the closed ball of radius R in $M^{m-1}(\kappa)$, so that

$$\mathrm{d}_{GH}(X, Y) < \varepsilon.$$

For the last case, we have

Theorem 5.9. Given an integer m, $\kappa \in \mathbb{R}$ and $R > 0$. If $\kappa > 0$, assume in addition that $R < \pi_\kappa/2$. For any $\varepsilon > 0$, there exists $\delta(\kappa, m, R) > 0$ such that the following holds: Let X^m be Alexandrov with curv $\geq \kappa$. Assume that $\overline{B_R(p)} = X$ for some $p \in \partial X$ and that

$$\mathcal{H}^{m-1}(\partial X) > \mathcal{H}^{m-1}(\partial B^\kappa_{R,+}) - \delta.$$

Then

$$\mathrm{d}_{GH}(X, \overline{B^\kappa_{R,+}}) < \varepsilon.$$

We give an outline of the proof.

Assume that we have a contradicting sequence (X^m_i, d_i) of curv $\geq \kappa$. This means that $\overline{B_R(p_i)} = X_i$ for some $p_i \in \partial X_i$ and $\mathcal{H}^{m-1}(\partial X_i) > \mathcal{H}^{m-1}(\partial B^\kappa_{R,+}) - \delta_i$, where $\delta_i \to 0$ as $i \to \infty$, but $\mathrm{d}_{GH}(X_i, \overline{B^\kappa_{R,+}}) > \epsilon$ for all i. By compactness, we may take a convergent subsequence,

which we also denote (X_i^m, d_i). As before, if we can rule out collapse then by the rigidity theorem 1.12 we will have a contradiction.

Assume that $(X_i, \mathrm{d}_i, p_i) \to (X_\infty, \mathrm{d}_\infty, p_\infty)$ and the convergence is collapsing. Then by the same argument as in the proof of the first part of Theorem 1.15, we have that $\dim X_\infty = m-1$ and $\partial X_i \to X_\infty$. Moreover, by the same fibration and covering argument we have that $\mathcal{H}^{m-1}(X_\infty) = \frac{1}{2}\mathcal{H}^{m-1}(\partial B_{R,+}^\kappa)$. Since $\overline{B_R(p_\infty)} = X_\infty$, by absolute volume comparison in dimension $m-1$, we have that $\mathcal{H}^{m-1}(X_\infty) \le \mathcal{H}^{m-1}(B_R^{\kappa,m-1})$, where $B^{\kappa,m-1}(R)$ is the R-ball in $M^\kappa(m-1)$. However, clearly $\frac{1}{2}\mathcal{H}^{m-1}(\partial B_{R,+}^\kappa) > \mathcal{H}^{m-1}(B_R^{\kappa,m-1})$ which is a contradiction.

Remark 5.10. The same argument as in the above proof can be used to rule out collapsing in the proof of Theorem 5.1 using that $\frac{1}{2}\mathcal{H}^{m-1}(\partial B_R^\kappa) > \mathcal{H}^{m-1}(B_R^{\kappa,m-1})$ when $R < \pi_\kappa/2$.

6. Boundary of Alexandrov spaces: background

In this section we collect various known results about the boundaries of Alexandrov spaces as well as some folklore results. We also include several new results.

We try to present all the results in the order they need to be proved to avoid logical loops in the arguments. For the folklore results we present proofs. We will stress topological results as they are less well-known and some are new. Throughout this section all Alexandrov spaces will be assumed to be finite dimensional.

Let X^n be an n-dimensional Alexandrov space of curv $\ge k$. Recall that the boundary ∂X of X is defined inductively as follows. For $n = 1$ it's known that X is a topological 1-manifold with (possibly empty) boundary and ∂X is defined as the manifold boundary of X. For $n > 1$ the boundary is defined inductively: $p \in \partial X$ iff the space of directions $\Sigma_p X$ has boundary. This definition makes sense since $\Sigma_p X$ is an Alexandrov space of curv ≥ 1 of dimension $n - 1$. Points which are not boundary points are called interior points. The set $X \setminus \partial X$ of all interior points is denoted by $\mathrm{Int}\, X$.

Definition 6.1. A metrizable space X is called an MCS-space (space with multiple conic singularities) of dimension n if every point $x \in X$ has a neighborhood which is pointed homeomorphic to an open cone over a compact $(n - 1)$-dimensional MCS-space. Here we assume the empty set to be the unique (-1)-dimensional MCS-space.

A compact 0-dimensional MCS-space is a finite collection of points with discrete topology and a 1-dimensional MCS-space is a graph.

By Perelman's Morse theory [Per93], an n-dimensional Alexandrov space X is an n-dimensional MCS space. This also follows by induction on dimension from Perelman's stability theorem [Per91, Kap07] which implies that a conical neighborhood of $x \in X$ is homeomorphic to $C(\Sigma_x X)$.

An open conical neighborhood of a point in an MCS-space is unique up to pointed homeomorphism [Kwu64]. As is true for MCS-spaces in general, X admits a canonical topological stratification. We say that a point $p \in X$ belongs to the l-dimensional strata X_l if l is the maximal number m such that the conical neighbourhood of p is pointed homeomorphic to $\mathbb{R}^m \times K(S)$ for some compact MCS-space S. It is clear that X_l is an l-dimensional topological manifold.

We will need the following key result of Perelman. We include its proof due to its importance.

Theorem 6.2. [Per91, Theorem 4.6] Let $\Sigma_1^{k_1}$ and $\Sigma_2^{k_2}$ be Alexandrov spaces of curv ≥ 1 such that $\mathbb{R}^{l_1} \times C(\Sigma_1) \overset{homeo}{\cong} \mathbb{R}^{l_2} \times C(\Sigma_2)$. Then Σ_1 has boundary iff Σ_2 does.

Proof. Note that we necessarily have that $l_1 + k_1 = l_2 + k_2$. Let's assume Σ_1 has boundary. We need to show that Σ_2 does too.

Let us first prove the result for $k_1 = 1$. If $k_1 = 1$ then $\Sigma_1 = I$ is a closed interval. We proceed by induction on k_2. If $k_2 = 1$ then Σ_2 can only be I or S^1 and since $\mathbb{R}^l \times C(I)$ is not homeomorphic to $\mathbb{R}^l \times C(S^1)$ the theorem is clear in this case.

Induction step on k_2: Let $k_2 > 1$ and suppose we have already proved the statement for smaller values of dim Σ_2.

Suppose $F \colon \mathbb{R}^{l_1} \times C(I) \to \mathbb{R}^{l_2} \times C(\Sigma_2^{k_2})$ is a homeomorphism and Σ_2 has no boundary.

Since $k_2 > 1 = k_1$ we must have $l_1 > l_2$. Therefore there is a point $x \in \mathbb{R}^{l_1} \times \{o_1\}$ whose image $y = F(x)$ does not lie in $\mathbb{R}^{l_2} \times \{o_2\}$, i.e., $y = (q, t, v_2)$ where $q \in \mathbb{R}^{l_2}, t > 0, v_2 \in \Sigma_2$. Since Σ_2 has no boundary $\Sigma_{v_2}(\Sigma_2)$ has no boundary either by definition. Then a conical neighborhood of y is homeomorphic to $\mathbb{R}^{l_2+1} \times C(\Sigma_{v_2}(\Sigma_2))$ by the stability theorem. By uniqueness of conical neighborhoods this implies that $\mathbb{R}^{l_1} \times C(I)$ is homeomorphic to $\mathbb{R}^{l_2+1} \times C(\Sigma_{v_2}(\Sigma_2))$. This contradicts the induction assumption on k_2.

This proves that if $\mathbb{R}^{l_1} \times C(I) \overset{homeo}{\cong} \mathbb{R}^{l_2} \times C(\Sigma_2)$ then Σ_2 has boundary.

Let us now do induction on k_1 in the main statement of the theorem. We have just verified the base of induction $k_1 = 1$.

Let $k_1 > 1$ and suppose the theorem is proved for smaller k_1. Suppose $F \colon \mathbb{R}^{l_1} \times C(\Sigma_1^{k_1}) \to \mathbb{R}^{l_2} \times C(\Sigma_2^{k_2})$ is a homeomorphism where Σ_1 has boundary but Σ_2 does not.

Let $v_1 \in \partial \Sigma_1$. Let $x = (0, 1, v_1) \in \mathbb{R}^{l_1} \times C(\Sigma_1)$ where $0 \in \mathbb{R}^{l_1}, 1 \in \mathbb{R}$. Let $y = F(x)$. We have two possibilities.

Case 1 $y = q \times o_2$ lies on R^{l_2} times the cone tip of $C(\Sigma_2)$. Then a conical neighborhood of x is homeomorphic to a conical neighborhood of y which gives that $\mathbb{R}^{l_1+1} \times C(\Sigma_{v_1}(\Sigma_1))$ is homeomorphic to $\mathbb{R}^{l_2} \times C(\Sigma_2^{k_2})$. Since $\Sigma_{v_1}(\Sigma_1)$ has boundary this contradicts the induction assumption.

Case 2 $y = (q, t, v_2)$ where $q \in \mathbb{R}^{l_2}, t > 0, v_2 \in \Sigma_2$. Then similarly to Case 1 we get that $\mathbb{R}^{l_1+1} \times C(\Sigma_{v_1}(\Sigma_1))$ is homeomorphic to $\mathbb{R}^{l_2+1} \times C(\Sigma_{v_2}(\Sigma_2))$, which is again a contradiction to the induction assumption since $\Sigma_{v_1}\Sigma_1$ has boundary while $\Sigma_{v_2}\Sigma_2$ does not. $\qquad\square$

Theorem 6.2 implies that boundary points are distinguished from interior points topologically and that a point in a conical neighborhood of an interior point cannot be a boundary point. This can be further sharpened (see Theorem 6.17 below) to say that boundary points have different local homology from interior points but in order to prove this the above result had to be proved first.

Theorem 6.2 immediately implies the following basic fact.

Corollary 6.3. [Per91, Theorem 4.6] Let X be Alexandrov. Then ∂X is a closed subset of X.

Theorem 6.2 also immediately implies that homeomorphisms between Alexandrov spaces preserve boundaries. Together with the stability theorem this implies that under noncollapsed convergence the boundaries converge to the boundary in the limit.

Theorem 6.4. Let $(X_i^m, p_i) \to (X^m, p)$ be a noncollapsing convergent sequence of Alexandrov spaces with curv $\geq \kappa$. Let $d_i = d_i(p_i, \partial X_i)$ (if $\partial X_i = \varnothing$ we set $d_i = +\infty$). Then X has nonempty boundary iff $d := \liminf d_i < \infty$. Moreover, if $d < \infty$ then $(X_i^m, \partial X_i, p_i) \to (X^m, \partial X_i, p)$.

Similar to Alexandrov spaces, for any $p \in \partial X$ there are two natural notions of the tangent space to the boundary $T_p \partial X$. One is the cone over $\partial \Sigma_p X$. Another is the limit of $(\partial X, p, \lambda d_X)$ as $\lambda \to \infty$.

Applying Theorem 6.4 to the convergence $(X, \lambda d, p) \to (T_p X, , d_{T_p X}, o_p)$ as $\lambda \to \infty$ gives that these two notions are the same. More precisely, we have

Corollary 6.5. [Per91] Let X be a Alexandrov space and let $p \in \partial X$. Then

$$(X, \partial X, \lambda d, p) \xrightarrow[\lambda \to \infty]{} (T_p X = C(\Sigma_p X), \partial C(\Sigma_p X) = C(\partial \Sigma_p X), d_{T_p X}, o_p).$$

Furthermore a small neighborhood of p in ∂X is homeomorphic to $C(\Sigma_p \partial X)$.

By induction on dimension in the last statement of the above theorem we get

Corollary 6.6. Let X^m be an m-dimensional Alexandrov space with nonempty boundary. Then ∂X is an $(n-1)$-dimensional MCS-space.

A key theorem in the theory of Alexandrov spaces is the following result of Perelman whose proof relies only on the stability theorem and the results stated above.

Theorem 6.7 (Doubling Theorem [Per91]). Let X be an Alexandrov space of curv $\geq \kappa$ with nonempty boundary. Then the doubling of X along the boundary is an Alexandrov space of curv $\geq \kappa$ without boundary.

By Morse theory, Theorem 6.2 immediately implies the following theorem.

Theorem 6.8. [PP93] For any Alexandrov space X its boundary ∂X is extremal in X.

Proof. Let $p \notin \partial X$. Let $q \in \partial X$ be such that $d(p, q) = d(p, \partial X)$. Let $f = d(\cdot, p)$. By [Pet07, Theorem 4.1.2], we need only to show that $\nabla_q f = 0$, i.e., q is a critical point of f. Suppose not. Then by Morse theory f is a local bundle map to $\mathbb{R}$ near q and hence there exists q' near q with $f(q') < f(q)$ such that a conical neighborhood of q is homeomorphic to a conical neighborhood of q'. But this is impossible by Theorem 6.2 since $q' \notin \partial X$ and hence $\Sigma_{q'} X$ has no boundary. $\qquad\square$

Remark 6.9. Theorem 6.8 gives a different proof of Corollary 6.6 using general results on topological structure of extremal subsets [PP93].

There is a third notion of the tangent space $T_p \partial X$ for $p \in \partial X$ where in the convergence of rescaled metrics on ∂X one looks at intrinsic rather than extrinsic metrics. This notion is generally different as should be expected. Nevertheless, one has the following.

Theorem 6.10. Let X^m be Alexandrov and $p \in \partial X$. Then $(\partial X, \lambda d_X^\partial, p) \to (T_p \partial X, d_{T_p X}^\partial, o_p)$ as $\lambda \to \infty$, where d^∂ stand for intrinsic metrics on the boundary on relevant spaces.

Proof. This follows from Corollary 6.5 along with the fact that that ∂X is extremal and a result of Petrunin stating that given noncollapsing convergence $(X^n, F_i, p_i) \to (X^n, F, p)$ where F_i are extremal in X, then the intrinsic metrics on F_i also Gromov-Hausdorff converge to the intrinsic metric on F. $\qquad\square$

Following Petrunin and Perelman [PP96] (see also [Pet07]) we can now adopt the following definition.

Definition 6.11. Let X be an Alexandrov space. A locally Lipschitz function $f\colon X \to \mathbb{R}$ is called *semiconcave* if for any any point in X has a neighborhood U such that its restriction of f to any geodesic in U is λ-concave for some $\lambda = \lambda(U)$.

f is called *double semiconcave* if it is semiconcave in the case where $\partial X = \varnothing$. In the case where $\partial X \neq \varnothing$ then we also require that the canonical extension of f to the double of X is semiconcave in the above sense.

Remark 6.12. Let us comment on the last requirement in the above definition. Much of the theory of semiconcave functions including most of the theory of their gradient flows can be developed without that assumption (see [AKP22]). However some results (such as Theorem 6.13 below) require existence of supporting vectors (see [Pet07, Definition 1.3.6]) to f at all points. Supporting vectors always exist at interior points of X but may not exist at boundary points without the doubling assumption [PP96]. Let us note however that existence of supporting vectors for distance functions, i.e., to functions of the form $\mathrm{d}(\cdot, A)$, outside A can be proved directly irrespective of the existence of the boundary (see [Per93, Proposition 2.2]). Therefore Theorem 6.13 for such functions can be proved directly without any reference to the notion of a boundary.

The following theorem of Petrunin and Perelman is very useful.

Theorem 6.13. Let X be an Alexandrov space and let $f\colon X \to \mathbb{R}$ be double semiconcave. Then the gradient flow of f leaves any extremal subset of X invariant.

Since boundary is extremal by Corollary 6.8 this immediately yields

Corollary 6.14. Let X be an Alexandrov space and let $f\colon X \to \mathbb{R}$ be double semiconcave. Then the gradient flow of f leaves ∂X invariant.

Let us remark here that Theorem 6.13 and Corollary 6.14 are false for semiconcave functions which are not double semiconcave. For example, $f = \mathrm{d}(\cdot, \partial X)$ is known to be semiconcave along geodesics in X [Per91, AB03], but its extension to the double of X is not semiconcave, supporting vectors to f do not exist at boundary points and the gradient flow of f clearly does not leave ∂X invariant.

The following result about the topology of the boundary is folklore but to the best of our knowledge the proof has not been carefully written. Therefore we record it here.

Proposition 6.15. Let X^n be an Alexandrov space. Then
  (1) ∂X is an $(n-1)$-dimensional MCS space;
  (2) $(\partial X)_{n-2} = \varnothing$;
  (3) For any $p \in \partial X$ it holds that $H_{n-2}(\Sigma_p \partial X, \mathbb{Z}_2) \neq 0$;
  (4) Let B be a compact connected component of ∂X. Then $H_{n-1}(B, \mathbb{Z}_2) \neq 0$.

Proof. Parts (1) and (2) follow from Corollary 6.5 by induction on dimension.

Using the fact that $(\partial X)_{n-2} = \varnothing$ parts (3) and (4) follow by [GP93, Lemma 1] which shows that a compact MCS space with empty codimension 1 strata has nontrivial top homology with $\mathbb{Z}_2$ coefficients.

$\square$

Remark 6.16. The statement that $(\partial X)_{n-2} = \varnothing$ can be interpreted as saying that the boundary of the boundary is empty.

We can now prove the following equivalent descriptions of boundary points. Recall that $\mathcal{S}^k(X)$ is defined as the set of points $p \in X$ such that $T_p X$ isometrically splits off $\mathbb{R}^k$ but not $\mathbb{R}^{k+1}$.

Theorem 6.17. Let X^n be an Alexandrov space and let $p \in X$. Then the following are equivalent

(1) $p \in \partial X$;
(2) $\Sigma_p(X)$ is contractible;
(3) $\tilde{H}_{n-1}(\Sigma_p X, \mathbb{Z}_2) = 0$;
(4) $H_n(X, X \setminus \{p\}, \mathbb{Z}_2) = 0$;
(5) $p \in \bar{X}_{n-1}$;
(6) $p \in \overline{\mathcal{S}^{n-1}(X)}$;

All results in this theorem are contained in the literature but we collect all the arguments in one place and give a proof that avoids circular reasoning.

Proof of Theorem 6.17. In what follows all homology is with coefficients in $\mathbb{Z}_2$. The theorem is trivial for $n = 1$. Therefore from now on we assume that $n \geq 2$.

First observe that by the stability theorem applied to the convergence $(X, \lambda \, \mathrm{d}_X, p) \xrightarrow[\lambda \to \infty]{}$ $(T_p X, \mathrm{d}_{T_p X}, o_p)$ we have that $H_n(X, X \setminus \{p\}) \cong H_n(T_p X, T_p X \setminus \{o_p\}) \cong H_{n-1}(\Sigma_p(X))$.

Let $\mathrm{Int}\, X = X \setminus \partial X$ be the interior of X. By induction on dimension using the stability theorem in the above convergence it follows that $(\mathrm{Int}\, X)_{n-1} = \varnothing$. Note that this means that $X_{n-1} \subset \partial X$.

Again using [GP93, Lemma 1] this implies that for any interior point $p \in \mathrm{Int}\, X$ it holds that $H_n(X, X \setminus \{p\}) \cong H_{n-1}(\Sigma_p X) \neq 0$.

If $p \in \partial X$ then Σ_p is a space of curv ≥ 1 with boundary. Since the distance function to the boundary on $\Sigma_p X$ is strictly concave its soul must be a point and hence $\Sigma_p X$ is contractible and $H_n(X, X \setminus \{p\}) \cong H_{n-1}(\Sigma_p) = 0$.

This establishes (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Let us establish (1) $\Leftrightarrow$ (5). We prove this by induction on n. The base case $n = 1$ is clear. Let $n > 1$ and suppose the equivalence (1) $\Leftrightarrow$ (5) has already been established for Alexandrov spaces of dimension less than n. We have already established that $X_{n-1} \subset \partial X$. Since ∂X is closed this implies that $\bar{X}_{n-1} \subset \partial X$.

Conversely, if $p \in \partial X$ then $\partial \Sigma_p \neq \varnothing$ and by the induction assumption Σ_p has nontrivial $n - 2$ strata. Therefore the vertex o lies in the closure of the $n - 1$ strata of $C(X)$ and by the stability theorem $p \in \bar{X}_{n-1}$.

This proves that $\partial X = \bar{X}_{n-1}$ which establishes (1) $\Leftrightarrow$ (5).

Let us establish (6) $\Leftrightarrow$ (1). By the splitting theorem it's obvious that if $p \in \mathcal{S}^{n-1}$ then $T_p X \cong \mathbb{R}_+^n$ and hence $p \in \partial X$. Since ∂X is closed it follows that $\overline{\mathcal{S}^{n-1}(X)} \subset \partial X$. The inverse inclusion follows from [Fuj18] since ∂X is an extremal subset of X. Alternatively it follows from the fact that if $x \in X$ is regular and $q \in \partial X$ satisfies $\mathrm{d}(x, q) = \mathrm{d}(x, \partial X)$ then $q \in \mathcal{S}^{n-1}$. This holds for example because in the double of X the point q lies on the interior of a geodesic connecting x with its mirror and hence q must be a regular point in the double by [Pet98].

$\square$

By the Stability Theorem part (2) in Theorem 6.17 implies that the punctured conical neighborhood of a point in ∂X is contractible. In a general n-dimensional MCS-space X the conical neighborhood of a point in X_{n-1} is homeomorphic to to $\mathbb{R}^{n-1} \times C(K)$ where K is a finite set of points and $|K| \neq 2$. If X^n is an n-dimensional Alexandrov space then $X_{n-1} \subset \partial X$ and hence the above shows that $|K|$ can only be equal to 1 since if $n \geq 2$ and $|K| \geq 3$ the punctured conical neighborhood of the apex in $\mathbb{R}^{n-1} \times C(K)$ is not contractible. In particular this means that X^n cannot contain a point whose conical neighborhood is homeomorphic to $\mathbb{R}^{n-1} \times C(K)$ where K is a finite set with $|K| \geq 3$.

Thus we get

Corollary 6.18. Let X be a n-dimensional Alexandrov space and $p \in X_{n-1}$. Then conical neighborhood of p is homeomorphic to $\mathbb{R}^n_+$.

For an n-dimensional Alexandrov space X^n without boundary Grove and Petersen proved in [GP93] that the top topological strata is connected and furthermore if X is compact then $H_n(X, \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is natural to ask if a similar result holds for connected components of the boundary. Grove and Petersen's proof uses that the set of regular points in an Alexandrov space is convex. However this is not known for connected components of the boundary. It would follow from the conjecture that the boundary is also an Alexandrov space but that conjecture remains wide open.

Nevertheless we show that the result of Grove and Petersen does hold for the boundaries of Alexandrov spaces. This sharpens Proposition 6.15.

Proposition 6.19. Let X^n be an n-dimensional Alexandrov space with $n \geq 2$. Then for every connected component Y of ∂X it holds that Y_{n-1} is connected and if Y is compact then $H_{n-1}(Y) \cong \mathbb{Z}_2$.

For the proof we need the following.

Lemma 6.20. Let X^n be Alexandrov of curv ≥ 1 such that $n \geq 2$ and $\partial X \neq \varnothing$.
Then ∂X is connected.

Proof. Suppose ∂X is not connected. Let p be the soul of Σ. By [PP96][Theorem 1.1] we have that $\Sigma_p X$ is homeomorphic to ∂X. Therefore $\Sigma_p X$ is not connected. Hence $T_p X$ has more than one end and contains a line. Therefore it splits a line isometrically by the splitting theorem and since $n \geq 2$ this implies that $\Sigma_p X$ is a spherical suspension over a nonempty set and hence is connected.

An alternative proof can be given by looking at the double Y of X along the boundary of X. If ∂X has more than one component then it is easy to see using that X is contractible that Y has infinite π_1. But Y is Alexandrov of curv ≥ 1 and hence must have finite π_1. $\square$

Proof of Proposition 6.19. It is enough to prove that Y_{n-1} is connected. Then for compact Y the isomorphism $H_{n-1}(Y) \cong \mathbb{Z}_2$ follows by [GP93, Lemma 1] using that $Y_{n-2} = \varnothing$.

We proceed by induction. The base of induction $n = 2$ is straightforward.

Induction step. Let $p \in Y$. By Corollary 6.5 a conical neighborhood U of p in Y is homeomorphic to $C(\partial \Sigma_p X)$.

By Lemma 6.20 $\partial \Sigma_p X$ is connected and by the induction assumption its top strata is connected. Therefore the same holds for U, that is, U_{n-1} is connected. Now the result trivially follows from connectedness of Y. $\qquad\square$

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