Homework, Chapter 1

This chapter serves as an introduction to the geometric way of looking at systems of differential equations. You should review your old ODE textbook, make sure you remember how to draw a phase portrait for an equation or a system, what is the difference between a solution curve and a trajectory, what are homogeneous systems, what is an equilibrium point.

Here are several practice problems which will help you to recall these concepts. Problems marked with * are more difficult.

1) For each of (a)-(d) below find an equation $\frac{dx}{dt} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases assume that $f(x)$ is a smooth function.)

(a) Every real number is an equilibrium point.
(b) Every integer is an equilibrium point, and there are no others.
(c) There are precisely three equilibrium points, all of them stable.
(d) There are no equilibrium points.

2*) Prove that a one-dimensional equation $\frac{dx}{dt} = f(x)$ cannot have nontrivial periodic solutions; that is a solution with $x(t) \equiv x(t+T)$ for some $T > 0$ must be constant.
Hint: Consider

$$\int_t^{t+T} f(x)\frac{dx}{dt}dt.$$

3) Consider a damped nonlinear oscillator

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k\sin x = 0, \text{ with } c > 0.$$

Prove that unless $(x(0), x'(0))$ is the equilibrium initial condition, the trajectory may never return to this point; that is

$$(x(t), x'(t)) \neq (x(0), x'(0)) \text{ for each } t > 0.$$

Hint: you do not need to solve the equation. Use the energy considerations in your reasoning.