

# APM 351: Differential Equations in Mathematical Physics

## Assignment 1, September 18 2009

### Summary

The general **first order linear PDE** in two variables has the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \quad (1)$$

The **method of characteristics** reduces this PDE to a system of coupled ODEs as follows:

(a) First determine the *characteristics* by solving

$$\frac{dx}{ds} = a(x, y), \quad \frac{dy}{ds} = b(x, y). \quad (2)$$

The solutions are called the **characteristic curves** of the PDE. If you wish, you can instead parametrize the characteristic curve over  $x$  and solve instead  $\frac{dy}{dx} = \frac{a(x, y)}{b(x, y)}$ .

(b) Along the characteristic curves, the PDE reduces to the ODE

$$\frac{d}{ds}u + cu = d, \quad (3)$$

where  $c = c(x(s), y(s))$ ,  $d = d(x(s), y(s))$  are the values of the coefficients along the characteristic curve. The solution gives the values of  $u(x(s), y(s))$  along the characteristic curve. If you have used instead  $x$  as the parameter, this ODE takes the form  $b \frac{d}{dx}u + cu = d$ , where  $b = b(x, y(x))$ , and correspondingly for  $c$ ,  $d$ , and  $u$ .

(c) To obtain the general solution of the PDE, assign initial values for  $u$  along some curve  $(x(t), y(t))$  by setting  $u(x(t), y(t)) = f(t)$ , where  $f$  is an arbitrary function. Make sure that the curve intersects each characteristic transversally and exactly once. If you have used  $x$  as parameter, you can use initial values  $u(0, y) = f(y)$ .

Given a point  $(x, y)$ , find the characteristic curve through  $(x, y)$  by solving (a) with  $x(0) = x$  and  $y(0) = y$ . Follow it to the point  $(x_0, y_0)$  where it intersects the initial curve, let  $s_0$  be the corresponding value of  $s$ , and read off the initial value  $u_0$ . Next solve the ODE (b) with initial value  $u(s_0) = u_0$ . The value of this solution at  $s = 0$  is the desired value of  $u(x, y)$ .

The method of characteristics can be extended to solve even the general first order PDE, at least in some neighborhood of the curve of initial values. But it can be tricky to apply. One of the reasons is that Eq. (2) is typically nonlinear, even when the PDE is linear, and hence its solutions may not be defined globally.

### Assignments:

In Chapter 2 of Strauss, read about the wave and diffusion equation.

## Hand-in (due Friday, September 25):

1. Use the method of characteristics to solve the initial-value problem for the transport equation

$$u_t + bu_x = f(x, t)$$

with initial values  $u(x, 0) = g(x)$ . Is the problem well-posed?

2. Solve  $2u_x + 3u_y + u = e^x$  with  $u(x, 0) = 0$ .
3. Solve the equation  $yu_x + xu_y = 0$  with initial values given by  $u(x, 0) = e^{-x^2}$ . In which region of the plane is the solution uniquely determined?
4. By trial and error, find a solution of the heat equation  $u_t = u_{xx}$  with initial condition  $u(x, 0) = x^2$ .
5. Find new variables  $\xi, \eta$  that transform the wave equation  $u_{tt} - c^2u_{xx} = 0$  into  $u_{\xi\eta} = 0$ . Use this to re-derive our formula for the general solution of the wave equation.
6. Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic (complex-analytic) function. Write  $z = x + iy$  and  $f(z) = u(x, y) + iv(x, y)$ , and interpret  $f$  as a function from  $\mathbb{R}^2$  to itself.
  - (a) The Cauchy-Riemann differential equations say that

$$u_x = v_y \quad u_y = -v_x.$$

Show that  $u$  and  $v$  satisfy Laplace's equation.

(b) Conversely, assume that  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies Laplace's equation. (We say that  $u$  is a **harmonic function**). Show that there exists a function  $v$  such that the Cauchy-Riemann differential equations hold. ( $v$  is called the **conjugate harmonic function** to  $u$ . The function  $u + iv$  is holomorphic.

7. If  $F$  is a continuous vector field on  $\mathbb{R}^3$  and  $|F(x)| \leq (1 + |x|^3)^{-1}$ , prove that

$$\int_{\mathbb{R}^3} \operatorname{div} F \, dx = 0.$$

*Hint:* Consider a large ball  $B_R$  and then take  $R \rightarrow \infty$ .