APM 351: Differential Equations in Mathematical Physics
Weekly Summary, October 9 2009

Summary

We turn to inhomogeneous equations and the role of boundary conditions for the heat and wave equation. We will formally write these equations as

$$u_t = Lu + f(\cdot, t)\,, \quad u(\cdot, 0) = \phi\,, \tag{1}$$

where L is a linear differential operator that involves only the x variable, f is the inhomogeneity or **source term**, and ϕ is the initial condition.

- To ground ourselves, let us first consider a linear ODE

  $$\frac{dy}{dt} = Ay + f(t) \quad y(0) = y_0\,, \tag{2}$$

  where A is a $n \times n$ matrix f is a given function, $y_0 \in \mathbb{R}^n$ a given vector, and the unknown function $y(t)$ takes values in $\mathbb{R}^n$. The **Variation of Constants formula** says that the solution is given by

  $$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)} f(s)\, ds\,.$$

  Here, e^{At} denotes the fundamental solution of the homogeneous equation $\frac{d}{dt}y = Ay$. By definition, $y(t) = e^{At}y_0$ solves the initial-value problem

  $$\frac{dy}{dt} = Ay \quad y(0) = y_0\,.$$

  There are many equivalent ways to define and compute the matrix-valued function e^{At}, by its power series, by diagonalizing A, or by special techniques such as contour integrals.

- Let us now try to solve the **heat equation with sources**

  $$u_t = ku_{xx}\,, \quad u(x, 0) = \phi(x)\,. \tag{3}$$

  Duhamel's formula says that the solution is given by

  $$u(\cdot, t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s) f(\cdot, s)\, ds\,.$$

  Here, $\mathcal{S}(t)$ is the fundamental solution of the homogeneous heat equation. Using the explicit formula from last week, we can spell out Duhamel's formula explicitly as

  $$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\, \phi(y)\, dy + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}}\, f(y,s)\, dyds\,.$$

- Consider finally the initial-value problem for the wave equation with sources,

$$u_{tt} = c^2 u_{xx} + f(x,t)\,, \quad u(x,u) = \phi(x), u_t(x,0) = \psi(x)\,. \tag{4}$$

  We rewrite this equation as a system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ c^2\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u(\cdot,0 \\ v(\cdot,0) \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

  and then apply Duhamel's formula

$$\begin{pmatrix} u(\cdot,t) \\ v(\cdot,t) \end{pmatrix} = \mathcal{T}(t)\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} 0 \\ f(\cdot,s) \end{pmatrix} ds\,.$$

  Here, $\mathcal{T}(s)$ is the fundamental solution of the above system. Using D'Alembert's formula, we obtain for the first component

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct)+\phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds\,.$$

We've given a recipe how to construct solutions of the inhomogeneous heat and wave equation. Our heuristic derivation does of course not constitute a proof. But once we guess the correct formulas, we can check by direct computation that they indeed satisfies Eqs. (3) and (4).

Assignments:

Read the second half of Chapter 2, specifically the last sections that contains the proof of the solution formula for the heat equation.

Correction to Assignment 2:

Problem 6b: For the diffusion equation $u_t = u_x x$ on $-1 < x < 1$ with Robin boundary conditions

$$u_x(-1,t) - au(-1,t) = u_x(1,t) + au(1,t) = 0\,,$$

we find that $u(t,x) = e^{-b^2 t}cos(bx)$ are solutions, if $a = b\tan b$. These solutions all decrease exponentially, and hence the energy also decreases exponentially.

For $a < 0$, there's another family of solutions, given by $u(x,t) = e^{b^2 t}\cosh(bx)$, where b should be chosen so that $a = -b\tanh(b)$. These solutions will grow exponentially, and so the energy also grows exponentially.