APM 351: Differential Equations in Mathematical Physics
Assignment 4, October 30 2009

Summary

(in preparation)

Assignments:

Read Chapter 5 of Strauss.

Hand-in (due Friday, November 6):

1. (a) On the interval $[-1, 1]$, show that the function x is orthogonal to the constant functions.
   (b) Find a quadratic polynomial that is orthogonal to both 1 and x.
   (c) Find a cubic polynomial that is orthogonal to all quadratics.
   (These are the first three Legendre polynomials.)

2. Let ϕ be a 2π-periodic function with Fourier series $\phi(x) = \sum_n A_n e^{inx}$.
   (a) If ϕ is real-valued, show that $A_{-n} = \bar{A}_n$.
   (b) If, additionally, ϕ is even, what can you say about the Fourier coefficients? Use this to represent ϕ as a cosine series.
   (c) What if ϕ is odd?

3. Given the Fourier sine series of the function $f(x) = x$ on $[0, \pi]$.
   (a) Apply Parseval's indentity to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
   (b) Integrate the sine series term by term to obtain a Fourier cosine series for the function $\frac{1}{2}x^2$. Note that the constant of integration appears as the $n = 0$ term in the series.
   (c) Then by setting $x = 0$, find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

4. Find the sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$.

5. (a) Expand the function $\phi(x) = |\sin x|$ as a cosine series on $[-\pi, \pi]$.
   (b) Find the sums $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$.

6. Let γ_n be a sequence of constants with $\lim_{n\to\infty} \gamma_n = \infty$. Define a sequence of functions on $[0, 1]$ by $f_n(x) = \gamma_n \sin(n\pi x)$ for $0 \le x \le \frac{1}{n}$, and $f(x) = 0$ otherwise.
   (a) Show that $f_n \to 0$ pointwise, but not unformly.
   (b) If $\gamma_n = n^{1/3}$, prove that $f_n \to 0$ in L^2.
   (c) If $\gamma_n = n^{2/3}$, show that f_n does not converge in L^2.

7. Let f be a smooth 2π-periodic function with $\int_0^{2\pi} f(x)\,dx = 0$. Use the Fourier series representation and Parseval's identity to show that $||f|| \le ||f'||$.

8. Let f be real-valued function on the real line. Assume that f is continuously differentiable, and that
   $$\left(\int_{\mathbb{R}} |f'(x)|^2 \, dx \right)^{\frac{1}{2}} = M < \infty \, .$$
   Use Schwarz' inequality to prove that $|f(y) - f(x)| \le M\sqrt{|y - x|}$. In particular, f is uniformly continuous.