APM 351: Differential Equations in Mathematical Physics
Assignment 5, November 11 2009

Summary

(in preparation)

Assignments:

Read Chapter 6 of Strauss.

Hand-in (due Friday, November 20):

1. Let D be an open set with smooth boundary in $\mathbb{R}^3$. Use the divergence theorem to show that the Dirichlet problem
$$\Delta u = f \text{ in } D\,, \quad u = g \text{ on } \partial D$$
cannot have a solution unless $\iiint_D f\,dxdydz = \iint_{\partial D} g\,dS$.

2. Find the radial solutions (depending only on $r = |x|$) of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u(r) = \frac{v(r)}{r}$. Solutions may blow up at $r = 0$.)

3. Solve $\Delta u = 0$ in the spherical shell $0 < a < r < b$ in $\mathbb{R}^n$ for $n \geq 2$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$. (*Hint:* Look for a radial solution. Your formula will look different for $n = 2$ than in higher dimensions).

4. Suppose that u is a harmonic function in the disk $D = \{r < 1\}$ in two dimensions, and that $u = 3 \sin 2\theta + 1$ for $r = 2$. Without computing the solution, find
(a) the maximum of u on D;
(b) the value of u at the origin.

5. Solve $u_{xx} + u_{yy} = 0$ in the disk $(r < a)$ with the boundary condition $u = \sin^3\theta$.
(*Hint:* Use the identity $\sin^3\theta = 3\sin\theta - 4\sin 3\theta$ and apply Poisson's formula.)

6. Derive Poisson's formula for the exterior of the unit disc $(r > 1)$.

7. Consider a homogeneous polynomial in two variables
$$P(x, y) = a_0 x^k + a_1 x^{k-1} y + \cdots + a_k y^k\,.$$
(a) Under what conditions on the coefficients is the polynomial harmonic? How many linearly independent harmonic polynomials are there of degree k?

   (b) Write down a basis of the space of harmonic polynomials of degree $k \leq 4$, in both Cartesian and polar coordinates. Identify them as the real (or imaginary) parts of holomorphic functions.

8. How many linearly independent polynomials of degree k are there in three variables? How many linearly independent *harmonic* polynomials of degree k are there? (*Hint:* Consider the Laplacian as a linear transformation that maps polynomials of degree k to polynomials of degree $k - 2$. You may assume that this map is onto.)

   Remark: The restriction of the harmonic polynomials to the unit sphere are called the **spherical harmonics**. They are widely used to solve rotationally symmetric problems in Theoretical Physics (you may have seen them in connection with angular momentum in Quantum Mechanics).