

# APM 351: Differential Equations in Mathematical Physics

## Assignment 6, January 11 2009

### Assignments:

Remind yourself of Chapter 7.4, and read Chapter 9 of Strauss.

### Summary:

We have studied the wave equation  $u_{tt} = c^2 \Delta u$  in two and three spatial dimensions. The initial conditions are given by  $u(x, 0) = \phi(x)$  and  $u_y(x, u) = \psi(x)$ . Two important property of the wave equation in any dimension are:

- The **energy**  $E(t) = \frac{1}{2} \int u_t(x, t)^2 + |\nabla u(x, t)|^2 dx$  is **conserved** (constant in time);
- **Causality:** The solution  $u(x, t)$  depends on the initial condition only inside the solid **light cone**

$$\{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid |y - x|^2 \leq |t - s|^2\}.$$

For the solution of the wave equation  $u_{tt} = c^2 \Delta u$  in three dimensions, we have derived **Kirchhoff's formula**

$$u(x, t) = \frac{1}{4\pi c^2 t^2} \int_{|y-x|=ct} \phi(y) + \nabla \phi(y) \cdot (y - x) + t\psi(y) dS(y).$$

Remarkably, the solution depends on the initial data only on the (surface of the) light cone, i.e., waves travel exactly at the speed of light. This is called **Huygens principle**. It is typical for solutions of the wave equation in all odd dimensions  $n = 2k + 1 \geq 3$ .

In two dimensions, we have **Poisson's formula**

$$u(x, t) = \frac{1}{4\pi c^2 t} \int_{|y-x|<ct} \frac{ct\phi(y) + ct\nabla\phi(y) \cdot (y-x) + ct^2\psi(y)}{(t^2 - |y-x|^2)^{1/2}} dy.$$

Note that Huygens' principle fails in two dimensions (and generally in even dimensions.)

### Hand-in (due Wednesday, January 20):

1. Find the Green's function of the upper unit half-ball

$$B_+ = \{x \in \mathbb{R}^3 \mid |x| < 1, x_3 > 0\}.$$

*Hint:* Start from the Green's function for the unit ball in Chapter 7.4 and use reflection.

2. A **plane wave** is a solution of the wave equation of the form  $u(x, t) = f(k \cdot x - ct)$ , where  $f$  is a  $C^2$ -function. Find all the three-dimensional plane waves.
3. Verify that  $u(x, t) = (c^2t^2 - |x|^2)^{-1}$  satisfies the three-dimensional wave equation except on the light cone.
4. (a) Derive the conservation of energy for the wave equation on a domain  $D$  with Dirichlet or Neumann boundary conditions.  
 (b) What about the Robin condition?
5. Consider the **Klein-Gordon equation**  $u_{tt} - c^2\Delta u + m^2u$ , where  $m > 0$ .  
 (a) What is the energy? Show that it is conserved.  
 (b) Prove the causality principle for it.
6. Thinking of space-time as  $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ , let  $\Gamma$  be the diagonal  $4 \times 4$  matrix with diagonal entries  $1, 1, 1, -1$ . A **Lorentz transformation** is an invertible matrix that satisfies  $L^t\Gamma L = \Gamma$ , or equivalently,  $L^{-1} = \Gamma L^t\Gamma$ .  
 (a) Prove that Lorentz transformations form a group, i.e., products and inverse of Lorentz transformations are again Lorentz transformations. What can you say about the determinant of  $L$ ?  
 (b) Show that  $L$  is Lorentz if and only if it preserves the quadratic form  $m(x, t) = |x|^2 - t^2$ , i.e.,  $m(L(v)) = m(v)$  for all  $v = (x, t) \in \mathbb{R}^4$ . The quadratic form  $m$  is called the **Lorentz metric**.  
 (b) If  $L$  is a Lorentz transformation, and  $U(z) = u(L(z))$ , show that
 
$$u_{tt} - \Delta u = U_{tt} - \Delta U,$$
 i.e., if  $u$  solves the wave equation, so does  $U$ .  
 (d) Explain the meaning of a Lorentz transformation in more geometrical terms. How does  $m$  relate to the light cone?
7. (a) Use Kirchhoff's formula to find the solution of the three-dimensional wave equation with initial data  $u(x, 0) = 0, u_t(x, 0) = x_2$ .  
 (b) Use the Euler-Poisson-Darboux equation to solve the three-dimensional wave equation with initial data  $u(x, 0) = 0, u_t(x, 0) = |x|^2$ .
8. (a) For any solution of the two-dimensional wave equation with initial data vanishing outside some circle, prove that  $u(x, t) = O(t^{-1})$  as  $t \rightarrow \infty$  for each fixed  $x \in \mathbb{R}^2$ , i.e.,  $tu(x, t)$  is bounded in  $t$  for each fixed  $x$ .  
 (b) Also show that  $\sup_x u(x, t) = O(t^{-1/2})$ , i.e.,  $t^{1/2}u(\cdot, t)$  is bounded *uniformly* in  $x$  as  $t \rightarrow \infty$ .