APM 351: Differential Equations in Mathematical Physics
Assignment 7, January 22 20010

Assignments:

Complete Chapter 9 of Strauss and move on into Chapter 10.

Summary:

For the wave equation $u_{tt} = c^2\Delta u$, the heat equation $u_t = k\Delta u$, and the Schrödinger equation $iu_t = -\Delta u$, separation of variables leads to the same eigenvalue problem

$$\Delta u = \lambda u\,.$$

It turns our that this eigenvalue problem has no solutions on $\mathbb{R}^n$ (except $u = 0$ for $\lambda = 0$) that decay at infinity or are even square integrable. (For every vector k, the function $u(x) = e^{-ik\cdot x}$ is a bounded solution with $\lambda = |k|^2$ but these don't lie in L^2.) So we have to investigate other methods of solutions.

- For the wave equation in two and three dimensions, the solution is given by the formulas of Poisson and Kirchhoff.
- The solution of the **heat equation** with $u(x, u) = \phi(x)$ is given by

  $$u(x,t) = (2\pi kt)^{-1}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}}\phi(y)\,dy\,.$$

  The positivity of the heat kernel $(2\pi kt)^{-1}e^{-\frac{|x|^2}{4kt}}$ is a manifestation of the maximum principle.

  This formula remains valid, if k is a complex number with positive real part, if we take the square root $\sqrt{k}$ to have positive real part. The integral converges and defines a smooth function, so long as ϕ is bounded and integrable.
- By analytic continuation to $k = i$, we obtain for the **Schrödinger equation** the solution formula

  $$u(x,t) = (2\pi it)^{-1}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}}\phi(y)\,dy\,.$$

  Here, the square root in the first factor should be chosen as $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. Note that the integral is now oscillatory, and will diverge unless ϕ itself decays at infinity. This is related to the wave-like properties of Schrödinger's equation. The fact that the kernel never vanishes indicates infinite speed of propagation.

Hand-in (due Friday, February 5):

1. Find a simple formula for the solution of the heat equation $u_t = \Delta u$ in three dimensions with initial values $u(x, y, z, 0) = xy^2z$.
   (*Hint:* Differentiate the equation and the initial values with respect to the variables x, y, z.)

2. Consider the one-dimensional wave equation $u_{xx} = c^2 u_{xx}$ with initial values given on a surface $\mathcal{S} = \{(x,t) \mid t = \gamma(x)\}$, by

$$u((x,\gamma(x)) = \phi(x)\,, \quad \frac{\partial u}{\partial n} = \Psi(x)\,.$$

If $\mathcal{S}$ is space-like, i.e., $|\gamma'(x)| < \frac{1}{c}$, prove that the initial-value problem has a unique solution. (*Hint:* The solution can be written as $u(x,t) = F(x+ct) + G(x-ct)$.)

3. (a) Verify that the solution formula for the heat equation in $\mathbb{R}^n$ is valid for every product of continuous functions $\phi(x) = \prod_{i=1}^n \phi_i(x_i)$, and hence for all finite linear combinations of such products.

   (b) Use an approximation argument to show that the formula holds more generally for every continuous function ϕ with compact support.

4. (a) Starting from the zeroth Hermite polynomial $H_0(x) = 1$, derive the first four Hermite polynomials from the recursion formula.

   (b) Show that all Hermite polynomials are given by $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$.

5. (a) Verify that the Hermite polynomials have the orthogonality property

$$\int H_k(x) H_\ell(x)\, e^{-|x|^2}\, dx = 0\,, \quad k \neq \ell\,.$$

   (b) Explain how to use the Gram-Schmidt method to obtain another recursion formula for the Hermite polynomials. (The resulting integrals can be computed explicitly, but you're not asked t do that here.)

6. Consider the eigenvalue problem $w'' - 2xw' + (\lambda - 1)w = 0$ that determines the Hermite polynomials.

   (a) Show that every solution with $\lambda \neq 2k+1$ is a power series but not a polynomial.

   (b) Deduce that for every such solution, $v(x) = w(x)e^{-\frac{x^2}{2}}$ grows rapidly as $|x| \to \infty$. (*Hint:* Use the recursion relation for the Taylor coefficients a_k of w as $k \to \infty$, and compare with the power series expansion for e^{x^2}.)

7. Solve the wave equation in the square $(0,\pi)\times(0,\pi)$ with homogeneous Neumann conditions on the boundary, and initial conditions $u(x,y,0) = \sin^2 x$.

8. Find all solutions of the wave equation of the form $u(x,t) = e^{-i\omega t} f(|x|,t)$ in two space dimensions.