MAT 1000 / 457 : Real Analysis I
Assignment 1, due September 19, 2012

1. *(Folland 1.1)* A family of sets $\mathcal{R} \in \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ and $E \setminus F \in \mathcal{R}$). A ring which is closed under countable unions is called a σ**-ring**.

   (a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

   (b) Let $\mathcal{R}$ be a ring. Then $\mathcal{R}$ is an algebra, if and only if $X \in \mathcal{R}$.

   (c) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R}$ or $E^c \in \mathcal{R}\}$ is a σ-algebra.

   (d) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}\}\}$ is a σ-algebra.

2. *(Folland 1.3)* Let $\mathcal{M}$ be an infinite σ-algebra. Show that ...

   (a) $\mathcal{M}$ contains an infinite sequence of disjoint non-empty sets;

   (b) $\mathcal{M}$ is uncountable.

3. *(Folland 1.4)* Let $\mathcal{A}$ be an algebra. Suppose that $\mathcal{A}$ is closed under countable increasing unions, i.e., $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ whenever $E_j \in \mathcal{A}$ and $E_j \subset E_{j+1}$ for each $j = 1, 2, \dots$.

   Prove that $\mathcal{A}$ is a σ-algebra.

4. *(Folland 1.8)* Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider a sequence $\{E_j\}_{j \geq 1}$ in $\mathcal{M}$. Define

$$\liminf E_j = \bigcup_{k \geq 1}^{\infty} \bigcap_{j=k}^{\infty} E_i \,, \qquad \limsup E_j = \bigcap_{k \geq 1}^{\infty} \bigcup_{j=k}^{\infty} E_i \,.$$

   (a) Show that

$$\begin{aligned} \liminf E_j &= \{x : x \in E_j \text{ for all but finitely many } j\} \,, \\ \limsup E_j &= \{x : x \in E_j \text{ for infinitely many } j\} \,. \end{aligned}$$

   Conclude that $\liminf E_j \subset \limsup E_j$.

   (b) Give an example of a sequence $\{E_j\}$ where $\liminf E_j \neq \limsup E_j$.

   (c) Show that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.
   If $\mu(\bigcup_{j=1}^{\infty} E_j) < \infty$, then also $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) *(Inclusion-Exclusion, Folland 1.9)*
   If $E, F \in \mathcal{M}$, then $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.

   (b) *(Restricting a measure to a subset, Folland 1.10)*
   Given a set $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Prove that μ_E is a measure.

6. *(summable* $\Rightarrow$ *countable*)
   Let Λ be an infinite set. For each $\lambda \in \Lambda$, let x_λ be a nonnegative number. Define the value of the series $\sum x_\lambda$ as the supremum of its finite partial sums,

$$\sum_{\lambda \in \Lambda} x_\lambda := \sup_{n \geq 0} \sup_{\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda} \{x_{\lambda_1} + \ldots + x_{\lambda_n}\} .$$

   (Note that the supremum is well-defined, even when its value is infinite.)

   If $\displaystyle\sum_{\lambda \in \Lambda} x_\lambda < \infty$, prove that $\Lambda' = \{\lambda \in \Lambda : x_\lambda > 0\}$ is countable.