MAT 1000 / 457 : Real Analysis I
Assignment 10, due Friday, December 7, 2012

1. *Stein and Shakarchi, Exercise 3.4)* Let f be an integrable function on $\mathbb{R}^d$ with $||f||_{L^1} = 1$.

   (a) Show that its maximal function satisfies

   $$Hf(x) \geq \frac{c}{|x|^d} \quad (|x| \geq 1)$$

   for some $c > 0$. Conclude that Hf is not integrable on $\mathbb{R}^d$.
   (*Hint:* Use that $\int_B |f| > 0$ for some ball B.)

   (b) Show that the weak-type estimate provided by the Hardy-Littlewood Maximal Theorem is best possible in the following sense: If f is supported in the unit ball, then

   $$m(\{x : Hf(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

   for some $c' > 0$ and all all sufficiently small $\alpha > 0$.

2. *(The volume of parallel sets depends continuously on the distance)*
   Let $K \subset \mathbb{R}^d$ be a non-empty compact set, let

   $$K_t = \{x \in \mathbb{R}^d : d(x, K) \leq t\} \quad (t > 0)\,,$$

   with the convention that $K_0 = K$, and consider $\rho(t) = m(K_t)$.

   (a) Show that ρ is right continuous (appeal to Problem 4 of Assignment 3).

   (b) Argue that, for $t > 0$, the level set $\{x : d(x, K) = t\}$ has measure zero.
   Hint: What can you say about the Lebesgue density of $\{x : d(x, K) < t\}$ at a point a with $d(a, K) = t$?

   (c) Conclude that ρ is also left continuous.

3. *(Stein and Shakarchi, Problem 3.1)* A collection $\mathcal{B}$ of balls is called a **Vitali covering** of a set E, if for every $x \in E$ and every $\delta > 0$ there is a ball $B \in \mathcal{B}$ such that $x \in B$ and $m(B) < \delta$.

   If E is a set of finite Lebesgue measure in $\mathbb{R}^n$, prove that for every $\varepsilon > 0$ there exists a disjoint collection $\{B_j\}_{j \geq 1}$ in $\mathcal{B}$ such that

   $$m\left(E \setminus \bigcup_{j=1}^{\infty} B_j\right) = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} m(B_j) \leq (1 + \varepsilon)\, m(E)\,.$$

   (*Remark:* The estimate remains valid for non-measurable sets, with Lebesgue measure replaced by outer Lebesgue measure.)

4. *(Folland 2.13)* Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions. Assume that $f_n \to f$ pointwise a.e., and that $\int f_n \to \int f$.

   (a) If f is integrable, show that $\int_E f = \lim \int_E f_n$ for all measurable sets E.

   (b) However, this need not be true if $\int f = \infty$.

5. *(Stein and Shakarchi, Exercise 2.18)* Let f be a real-valued function on $[0, 1]$, and suppose that $|f(x) - f(y)|$ is integrable on $[0, 1] \times [0, 1]$. Show that f is integrable on $[0, 1]$.

6. *(Folland 3.41)* Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval $I \subset [0, 1]$ of positive length (as constructed in Problem 6 of Assignment 4).

   (a) Let $F(x) = m([0, x] \cap A)$. Then F is absolutely continuous and strictly increasing, but F' vanishes on a set of positive measure.

   (b) Let $G(x) = m([0, x] \cap A) - m([0, x] \setminus A)$. Then G is absolutely continuous, but not monotone on any subinterval $I \subset [0, 1]$.
   (*Remark:* Compare with Problem 3 of Assignment 8)