# MAT 1000 / 457 : Real Analysis I
# Assignment 2, due September 26, 2012

1. (a) If $\mu : \mathcal{P}(\mathbb{R}) \to \{0, 1\}$ is a measure that takes only the values $0$ and $1$, prove that either $\mu = 0$, or else $\mu = \delta_a$ for some $a \in \mathbb{R}$.
   *Hint:* Recall the proof of the Bolzano-Weierstrass theorem.

   (b) What can you say if $\mu$ is only finitely additive?

2. *(Limits of measures are measures)*
   Let $(\mu_n)_{n\geq 1}$ be a sequence of measures on a measurable space $(X, \mathcal{M})$. Suppose that
   $$\mu(A) := \lim_{n\to\infty} \mu_n(A) < \infty$$
   exists for each $A \in \mathcal{M}$. Prove that $\mu$ is a measure.

3. *(The Borel-Cantelli lemma)*
   Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $E_j)_{j\geq 1}$ is a sequence of measurable sets with the property that
   $$\sum_{j=1}^{\infty} \mu(E_j) < \infty\,,$$
   and let $\limsup E_j = \{x : x \in E_j \text{ for infinitely many } k\}$, as in Assignment 1.
   Show that $\mu(\limsup E_j) = 0$.

4. *(Intermediate value theorem for measures).*
   Let $(X, \mathcal{M}, \mu)$ be a measure space with the property that for every set $A \in \mathcal{M}$ of positive measure there exists a subset $B \subset A$ with $0 < \mu(B) < \mu(A)$. Prove that $\mu(\mathcal{M}) = [0, \mu(X)]$, i.e., $\mu$ takes on all possible values up to to $\mu(X)$.

5. *(Unions over chains of closed sets)*
   Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ be a collection of closed subsets of the real line, with the property that for each pair of sets $A, B \in \mathcal{C}$, either $A \subset B$ or $B \subset A$. Prove that $\bigcup_{A\in\mathcal{C}} A$ is a Borel set. (*Hint:* Rewrite the union as a countable union.)

6. *(The section property of Borel sets)*
   Let $A$ be a Borel set in $\mathbb{R}^2$. Prove that, for every $y \in \mathbb{R}$, the cross section
   $$A(y) = \{x \in \mathbb{R} : (x, y) \in A\}$$
   is a Borel set. (*Hint:* Try using the Monotone Class Theorem.)