MAT 1000 / 457 : Real Analysis I
Assignment 3, due October 3, 2012

1. Let μ^* be an outer measure on X induced from a premeasure μ_0, and let μ be the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets.

   (a) *(Folland 1.18)* Prove that $\mu^*(A) = \inf_{E \in \mathcal{M}: E \supset A} \mu(E)$ for all $A \subset X$.

   (b) *(Folland 1.19)* If $\mu_0(X) < \infty$, define the *inner measure* of a set $A \subset X$ by $\mu_*(A) = \mu_0(X) - \mu^*(A^c)$. Prove that A is measurable, if and only if $\mu^*(A) = \mu_*(A)$.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive measure.

   (a) *(Folland 1.30)*
   Prove thay for every $\alpha < 1$ there is an open interval I such that $m(E \cap I) \geq \alpha m(I)$.

   (b) *(Folland 1.31)*
   Conclude that the set $E - E := \{x - y : x, y \in E\}$ contains an interval centered at 0.
   (*Hint:* Use (a) with $\alpha > 3/4$.)

3. Let $N \subset \mathbb{R}$ be a set of Lebesgue measure zero. Prove that there exists $c \in \mathbb{R}$ such that the translated set $c + N := \{c + x : x \in N\}$ contains no rational point.

   Hint: Consider the union $\bigcup_{q \in \mathbb{Q}} \{c \in \mathbb{R} : q \in c + N\}$.

4. *(Stein & Shakarchi 1.5)* Given a set $E \subset \mathbb{R}^n$, consider the open sets
$$U_n = \left\{x \in \mathbb{R}^n : d(x, E) < \tfrac{1}{n}\right\} .$$
   (a) If E is compact, prove that $m(E) = \lim m(U_n)$.

   (b) Give an example of a bounded open set where the conclusion fails.

5. Let $(f_n)_{n \geq 1}$ be a sequence of continuous real-valued functions on an interval $[a, b]$.
   Prove that
$$E = \left\{x : \sum_{n \geq 1} f_n(x) \text{ converges}\right\}$$
   is a Borel set. (*Hint:* Use the Cauchy criterion to show that E is an $F_{\sigma\delta}$-set.)

6. Does there exist a dense subset of $\mathbb{R}^2$ such that no three points are collinear?