MAT 1000 / 457 : Real Analysis I
Assignment 4, due October 10, 2012

1. *(Folland 2.14)*
   Let f be a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$. For $E \in \mathcal{M}$, set $\lambda(E) = \int_E f\, d\mu$. Show that ...

   (a) ... λ is a measure;

   (b) ... $\int g\, d\lambda = \int fg\, d\mu$ for every nonnegative measurable function g.

2. Let $x \in (0,1)$, and let $(x_i)_{i\geq 1}$ be its decimal expansion.
   (If x has several decimal expansions, use the one that terminates in 0.)

   (a) Show that
   $$f(x) = \limsup_{n\to\infty} \left(\frac{1}{n} \#\{i = 1, \dots, n : x_i = 7\} \right)$$
   defines a Borel measurable function on the unit interval.

   (b) Show that f assumes every value in $[0,1]$ on each nonempty subinterval $(a,b) \subset (0,1)$.

   (c) Construct a Borel measurable function that assumes every value in $[-\infty, \infty]$ on each nonempty subinterval of $(0,1)$.

3. Let $(f_n)_{n\geq 1}$ be a sequence of measurable real-valued functions on $\mathbb{R}$. Prove that there exist constants $c_n > 0$ such that the series $\sum c_n f_n(x)$ converges for almost every $x \in \mathbb{R}$.
   (*Hint:* Borel-Cantelli.)

4. *(Folland 2.9)*
   Let f be the Cantor-Lebesgue function (the 'devil's staircase') from Section 1.5, and let $g : [0,1] \to [0,2]$ be defined by $g(x) = f(x) + x$. Prove the following assertions.

   (a) g maps $[0,1]$ bijectively onto $[0,2]$, and $h = g^{-1}$ is continuous.

   (b) If C is the Cantor set, then $m(g(C)) = 1$.

   (c) Let $A \subset g(C)$ be a nonmeasurable set. Then $B := g^{-1}(A)$ is Lebesgue measurable, but not Borel. Hence $\mathcal{X}_A = \mathcal{X}_B \circ h$ is not Lebesgue measurable.
   (*Remark:* You may take for granted that every set of positive Lebesgue measure contains a nonmeasurable subset, see Folland Problem 1.29).

5. (a) *(Inclusion-exclusion)*
Let $(X, \mathcal{M}, \mu)$ be a measure space. If $A_1, \ldots, A_n$ are sets of finite measure in $\mathcal{M}$, prove that

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq F \subset \{1,\ldots,n\}} (-1)^{\#F+1} \mu\left(\bigcap_{i \in F} A_i\right) .$$

(*Hint:* Use characteristic functions.)

(b) *(The forgetful secretary)*
Let S_n be the set of all permutations of $\{1, \ldots, n\}$, and let μ_n be the normalized counting measure defined by $\mu_n(A) = (\#A)/(\#S_n)$. Find

$$p_n = \mu_n(\{\pi \in S_n : \pi(i) \neq i \text{ for } i = 1, \ldots, n\}) ,$$

and compute $p = \lim p_n$. (*Hint:* Consider $A_i = \{\pi \in S_n : \pi(i) = i\}$.)

6. *(Folland 1.33)*
Construct a Borel set A such that $0 < m(A \cap I) < m(I)$ for every subinterval $I \subset [0, 1]$. (*Hint:* Every interval contains Cantor-type sets of positive measure.)