MAT 1000 / 457 : Real Analysis I
Assignment 5, due October 17, 2012

1. *(Folland 2.16)*
   If f is a nonnegative integrable function, then, for every $\varepsilon > 0$ there exists a set E of finite measure such that $\int_E f > (\int f) - \varepsilon$.

2. Let f be an integrable function with the property that $\int_E f \geq 0$ for every measurable set E. Prove that f is nonnegative. In particular, if $||f||_{L^1} = 0$ then $f = 0$ a.e.

3. *(Folland 2.17)*
   Assume Fatou's lemma and deduce the Monotone Convergence Theorem from it.

4. *(Error term in Fatou's lemma)*
   Let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions that converges pointwise a.e. to f.

   (a) Prove that
   $$\lim_{n\to\infty} \left\{ \int |f_n| - \int |f - f_n| \right\} = \int |f(x)| \,.$$

   (b) Show that this implies Fatou's lemma.

5. *(Derivatives are measurable)*
   Recall that a real-valued function $f : (a, b) \to \mathbb{R}$ is differentiable, if the limit
   $$f'(x) = \lim_{y\to x, y\neq x} \frac{f(y) - f(x)}{y - x}$$
   exists for each $x \in (a, b)$. If f is differentiable, prove that f and f' are Borel measurable.

6. *(The Nikodym distance)* Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

   (a) Verify that $d(A, B) = \mu(A \triangle B)$ defines a metric on $\mathcal{M}/\sim$ with a suitable equivalence relation $\sim$. (Be brief!)

   (b) Prove that the metric space is complete.

   Remark: Please work directly with the measures, without referring to L^1 or the Dominated Convergence Theorem.