MAT 1000 / 457 : Real Analysis I
Assignment 6, due October 24, 2012

1. Find a simple (useful) condition that guarantees that
$$\sum_{n=1}^{\infty}\left(\int f_n\,d\mu\right) = \int\left(\sum_{n=1}^{\infty} f_n\right)\,d\mu\,.$$

2. Let $(f_n)_{n\geq 1}$ be a sequence of functions on $[0,1]$ that is bounded in L^2 (i.e., $\sup_n ||f_n||_2 < \infty$). Assume that there exists a measurable function f such that
$$\int_0^1 |f_n - f|\,dm = 0 \qquad (n\to\infty)\,.$$
Show that $f\in L^2$. Does it follow that $f_n \to f$ in L^2 ?

3. *(Folland 6.5)* Let $(X,\mathcal{M},\mu)$ be a measure space, and $1\leq p<q<\infty$. Show that ...
   (a) ... $L^p \not\subset L^q$ if and only if X contains sets of arbitrarily small measure;
   (b) ... $L^q \not\subset L^p$ if and only if X contains arbitrarily large finite measure.
   (c) What about the case $q=\infty$?

4. *(Folland Theorem 6.8)* Let $(X,\mathcal{M},\mu)$ be a measure space. Prove that ...
   (a) ... L^∞ is complete;
   (b) ... the bounded simple functions are dense in L^∞.

5. Are the continuous functions dense in $L^\infty(\mathbb{R})$? Is translation $t\mapsto f(\cdot - t)$ continuous?

6. *(Hausdorff measure and dimension)*
   Let X be a metric space with distance function $d(x,y)$. For $A\subset X$, the quantity $\operatorname{diam} A = \sup_{x,y\in A} d(x,y)$ is called the *diameter* of A. Define, for $s\geq 0$ and $A\subset X$
$$H_s(A) = \lim_{\delta\to 0}\left(\inf\left\{\sum_{j=1}^{\infty}(\operatorname{diam} B_j)^s : A\subset\bigcup_{j=1}^{\infty} B_j \text{ and } \mathrm{diam\,B_j}\leq\delta\right\}\right)\,.$$
   (a) Prove that the limit exists and defines an outer measure on X.

   (b) Let $0\leq s<t<\infty$. If $H_s(A)<\infty$ for some $A\subset X$, show that $H_t(A)=0$. Similarly, if $H_t(C)>0$ for some $C\subset X$, show that $H_s(C)=\infty$.

   (c) The *Hausdorff dimension* of a set $A\subset X$ is defined by $\dim_H(A)=\inf\{s : H_s(A)=0\}$. Compute the Hausdorff dimension of the standard Cantor set.

   Remark: Clearly, $H_s(A\cup B) = H_s(A)+H_s(B)$, if the sets A and B are separated by $\operatorname{dist}(A,B)=\inf_{x\in A,y\in B} d(x,y)>0$. This property implies that Borel sets are measurable for H_s (see Folland Chapter 11.2.) By Carathéodory's theorem, H_s defines a Borel measure, called the s-dimensional Hausdorff measure.