MAT 1000 / 457 : Real Analysis I
Assignment 7, due October 31, 2012

1. *(Folland 2.28d)* For $a \in \mathbb{R}$, compute

$$\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\,dx\,,$$

and justify the calculations. (*Remark:* The answer depends on the sign of a. How does it accord with the various convergence theorems?)

2. *(Folland 2.48)* Let $\mu = \nu$ be counting measure on $\mathbb{N}$. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Then $\iint f\,d\mu d\nu$ and $\iint f\,d\nu d\mu$ exist and are unequal, while $\int |f|\,d(\mu\times\nu) = \infty$.

3. *(Folland 2.60: The Beta-integral)*
The Gamma-function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$. Show that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (x,y>0)\,.$$

Hint: Write $\Gamma(x)\Gamma(y)$ as double integral and change variables to simplify the exponential.

4. Let M be a positive definite symmetric $n\times n$ matrix. Find the measure of the ellipsoid

$$E = \{x\in\mathbb{R}^n : x\cdot Mx < 1\}$$

in terms of M and the measure of the unit ball. (*Hint:* Diagonalize M.)

5. *(Folland 2.59: The Dirichlet integral)* Show that

$$\lim_{b\to\infty}\int_0^b x^{-1}\sin x\,dx = \frac{\pi}{2}\,.$$

Hint: Integrate the function $e^{-xy}\sin x$ with respect to x and y, and use that

$$\int e^{-xy}\sin x\,dx = -e^{-xy}\big(\tfrac{1}{1+y^2}\cos x + \tfrac{y}{1+y^2}\sin x\big)\,.$$

Please be careful ... the function $f(x) = x^{-1}\sin x$ is not integrable over $(0,\infty)$!

6. *(The Riemann-Lebesgue lemma)* If f is integrable on $\mathbb{R}$, prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)e^{-inx}\,dx = 0\,.$$