MAT 1000 / 457 : Real Analysis I
Assignment 8, due November 21, 2012

1. *(Folland 6.38)* Let f be a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$. Prove that
$$f \in L^p \iff \sum_{k=-\infty}^{\infty} 2^{kp}\mu(\{x : f(x) > 2^k\}) < \infty\,.$$

2. *(Folland 2.61)* If f is continuous in $[0, \infty)$, for$\alpha > 0$ and $x \geq 0$ let
$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt\,.$$
$I_\alpha f$ is called the α^{th} **fractional integral** of f.

   (a) Prove that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$.
   Hint: Use Problem 3 from Assignment 7 / Folland 2.60.

   (b) If $n \in \mathbb{N}$, then $I_n f$ is an n-th order antiderivative of f.

3. *(Existence of nowhere monotone continuous functions.)*
Prove that there exists a function in $\mathcal{C}([0,1])$ that is not monotone on any subinterval of positive length.

   Hint: Given a closed subinterval $I \subset [0,1]$ of positive length, prove that the set
$$A_I = \left\{ f \in \mathcal{C}([0,1]) : f\big|_I \text{ is monotone} \right\}$$
is closed and contains no open balls, and then apply the Baire Category Theorem. [1] (You may find it useful that the piecewise linear functions form a dense subspace of $\mathcal{C}([0,1])$.)

4. For $1 \leq k \leq n$, compute the spherical integral
$$\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} (u_1^2 + \cdots + u_k^2)^{-1/2}\, d\sigma(u)\,,$$
where σ is the standard rotationally invariant surface measure on $\mathbb{S}^{n-1}$.

   Hint: Rewrite this as a Gaussian integral over $\mathbb{R}^n$. Write your answer either in terms of the Gamma-function or in terms of the measures ω_d of the d-dimensional unit balls.

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[1] Just to fix the language:

- A **meager** subset of a topological space X is the countable union of nowhere dense sets;
- a **residual** set in X is a subset whose complement is meager;
- we say that **"a typical point of X has property φ"** if the set of all $x \in X$ with φ is residual.

The Baire Category Theorem states that a residual set in a complete metric space is not empty (i.e., the entire space is not meager). It is a (non-constructive) tool for proving existence theorems.

5. Let $\{f_n\}_{n\geq 1}$, f, g be functions in $L^2[0,1]$, with $f_n \to f$ pointwise a.e.

(a) If $|f_n(x)| < |x|^{-\frac{1}{3}}$, prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\,dx = \int_0^1 f(x)g(x)\,dx\,.$$

(b) If, instead, $||f_n||_{L^2} \leq M$ for all n and g is bounded, then the same conclusion holds.

6. *(The bathtub principle, Lieb & Loss Theorem 1.14)*
Let $(X, \mathcal{M}, \mu)$ be a measure space, and let V be a real-valued measurable function on X such that the sub-level sets $S_t := \{x : V(x) < t\}$ have finite measure for each $t \in \mathbb{R}$. Given $M > 0$, consider the problem of minimizing

$$I(g) = \int V(x)g(x)\,d\mu$$

among all functions g with $0 \leq g \leq 1$ and $\int g = M$.

(a) Prove that the minimum is assumed by the characteristic function of some set $A \subset \mathcal{M}$.
Hint: Try $A = S_t$ for a suitable choice of t. A sketch will help ...

(b) Describe all possible minimizers. Under what conditions on M is the minimizer unique (up to a set of measure zero)?