MAT 1000 / 457 : Real Analysis I
Assignment 9, due November 28, 2012

1. *(Folland 3.7)* Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Show that, for every measurable set $E \subset X$,

   (a) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$, and correspondingly for ν^-;

   (b) the total variation measure $|\nu| = \nu^+ + \nu^-$ satisfies

$$|\nu|(E) = \sup\left\{\sum_{j=1}^{n} |\nu(E_j)| : n \geq 0, E_1, \ldots, E_n \subset E \text{ disjoint, and} \bigcup_{j=1}^{n} E_j = E\right\}.$$

2. (a) Let A, B be non-empty open sets in $\mathbb{R}^n$, and let $T : A \to B$ be a bijection. Assume that for subsets $E \subset A$, the image $T(E)$ is (Lebesgue) measurable if and only if E is measurable. Prove that the **pushforward** of Lebesgue measure to B, given by

$$T\#m(F) = m(T^{-1}(F)), \quad \text{for } F \subset B$$

   is absolutely continuous with respect to Lebesgue measure.
   Hint: Use that sets of positive measure contain non-measurable subsets.

   (b) If T is a diffeomorphism, find the density of $T\#m$ with respect to Lebesgue measure, i.e., find a function f such that $d(T\#m) = f dm$.

3. *(Lebesgue measure = n-dimensional Hausdorff measure)*

   (a) Prove that the n-dimensional Hausdorff measure of an n-dimensional cube is finite and positive.

   (b) Conclude that Hausdorff measure agrees with Lebesgue measure up to a constant factor. What is the value of the constant?

   Hint: Use translations and dilations. To simplify the argument, modify the definition of Hausdorff measure to use coverings by balls,

$$H_n(A) = \lim_{\delta \to 0} \left(\inf \left\{ \sum_{j=1}^{\infty} r_j^n : A \subset \bigcup_{i=1}^{\infty} B_{r_j}(a_j), r_i < \delta, a_j \in X \text{ for } j = 1, 2, \ldots \right\} \right).$$

4. Show that $\mathbb{Q}$ is an F_σ but not a G_δ-subset of $\mathbb{R}$. (*Hint:* Use the Baire Category Theorem.)

   Remark: This implies that the irrationals $\mathbb{R} \setminus \mathbb{Q}$ cannot be written as a union of countably many compact sets. Also, there is no measurable function f that is continuous precisely on

$\mathbb{Q}$, because $C = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$ is a G_δ-set. To see this, note that the C is the zero set of the **oscillation**

$$\operatorname{osc}_f(x) = \lim_{r\to 0} \sup_{y,z\in(x-r,x+r)} |f(y) - f(z)| \,.$$

5. *(Folland 3.17)* Let $(X, \mathcal{M}, \mu)$ be a probability space, and let ν be the restriction of μ to a sub-σ-algebra $\mathcal{N} \subset \mathcal{M}$.

   (a) Given $f \in L^1(\mu)$, show that there exists a unique $g \in L^1(\nu)$ such that

   $$\int_A f\, d\mu = \int_A g\, d\nu$$

   for all $A \in \mathcal{N}$.

   Remark: In probability theory, g is called the **conditional expectation** of f on $\mathcal{N}$ and denoted by $g = E(f|\mathcal{N})$. Note that we can consider $E(f|\mathcal{N})$ as an element of $L^1(\mu)$; also, if f itself happens to be $\mathcal{N}$-measurable, then $E(f|\mathcal{N}) = f$. In this notation, setting $A = X$, we obtain the **conditional averaging** formula

   $$E(f) = E(E(f|\mathcal{N}))\,.$$

   (b) For example, on $\mathbb{R}$ with Lebesgue measure, compute $g = E(f|\mathcal{N})$ where

   $$\mathcal{N} = \{A \in \mathcal{M} : x \in A \Leftrightarrow -x \in A\}$$

   is the σ-algebra of symmetric measurable sets.

   (c) If $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ is the product σ-algebra on a product space $X = X_1 \times X_2$, where $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ spaces, and $\mathcal{N} = \mathcal{M}_1 \otimes \{0, X_2\}$ is the σ-algebra of vertical strips, find a formula for $E(f|\mathcal{N})$.

   Hint: Think of the Fubini-Tonelli theorem.

6. *(Hausdorff metric vs. Nikodym metric)* Let K be a compact set. For $\delta > 0$, the sets

   $$K_\delta = \{x \in \mathbb{R}^n : d(x, K) \le \delta\}$$

   are called the **(outer) parallel sets** of K. Prove that a sequence of non-empty compact sets $(K_j)_{j\ge1}$ converges to a compact set L in Hausdorff metric, if and only if the parallel sets of K_j converge in symmetric difference to the corresponding parallel sets of L, i.e.,

   $$\lim_{j\to\infty} m((K_j)_\delta \triangle L_\delta) = 0$$

   for all $\delta > 0$.

   Remark: But note that the two metrics are not comparable: Finite sets are dense in the space of compact sets (with the Hausdorff metric); in particular, every compact set of positive measure is the Hausdorff limit of finite sets (having measure zero). Worse, a typical compact set has Lebesgue measure zero by the Baire Category Theorem. On the other hand, the Hausdorff distance between two distinct non-empty compact sets of measure zero is strictly positive.