MAT 1000 / 457 : Real Analysis I
Final Exam, December 14, 2011

(Six problems; 20 points each. Time: 3 hours. No aids allowed)

Please be brief but justify your answers, citing relevant theorems. Sometimes a sketch can help!

1. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ such that $\int_0^1 |f_n|^2\, dm \le M$ for all n. Assume that there exists a measurable function f such that

$$\int_0^1 |f_n - f|\, dm \to 0 \qquad (n \to \infty)\,.$$

   (a) Show that $\int_0^1 |f|^2\, dm \le M$.

   (b) Does it follow that

$$\int_0^1 |f_n - f|^2\, dm \to 0 \qquad (n \to \infty)\,?$$

2. Let $\{A_n\}$ be an increasing sequence of (not necessarily measurable) sets in $\mathbb{R}^d$, i.e., $A_n \subset A_{n+1}$ for all n, and let A denote their union. Prove that their outer measure satisfies

$$m_*(A_n) \to m_*(A) \quad (n \to \infty)\,.$$

   Hint: Construct an increasing sequence of measurable sets $G_n \supset A_n$ with $m(G_n) = m_*(A_n)$.

3. State ...

   (a) ... the Brunn-Minkowski inequality;

   (b) ... the Change of Variables formula for integrals in d dimensions;

   (c) ... the Hardy-Littlewood maximal function theorem.

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4. Let $K(x,y)$ be a measurable complex-valued function on $\mathbb{R}^2$ with

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)|^2\,dxdy < \infty\,.$$

(a) Prove that if $f \in L^2(\mathbb{R})$, then the integral

$$Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy$$

converges for a.e. $x \in \mathbb{R}$.

(b) Show that $f \mapsto Tf$ defines a bounded linear transformation from $L^2(\mathbb{R})$ to itself.

(c) What is the adjoint of T?

5. True or False? Why? (Try to find one-line answers.)

(a) If $\{e_n\}_{n\geq 1}$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then

$$\langle f, e_n\rangle \to 0 \quad (n\to\infty)$$

for every $f \in \mathcal{H}$.

(b) If a measurable set $E \subset [0,1]$ satisfies

$$m(E\cap I) \geq \frac{1}{2}m(I)$$

for every $I \subset [0,1]$, then $m(E) = 1$.

6. Let F be a convex function of a single variable, i.e.,

$$F((1-s)x + sy) \leq (1-s)F(x) + sF(y)\,, \quad \text{for all } x,y \in \mathbb{R}, s\in[0,1]\,.$$

Prove that F is absolutely continuous on finite intervals.
Hint: Argue that the difference quotient

$$Q(a,b) = \frac{F(b)-F(a)}{b-a}\,, \quad (a<b)$$

is increasing in both a and b, and consider the one-sided derivatives D_+F and D_-F. What can you say about the integrals $\int_a^x D_\pm F(t)\,dt$?

— *end* —