Practice Problems (collected at UVa)

1. Define ...

   measure, outer measure, σ-algebra, complete measure; Borel set, Lebesgue measurable set; integrable function; L^p-norms and spaces, essential supremum; outer regularity; maximal function, density point of a set, Lebesgue point of a function, Lebesgue-Stieltjes measures on $\mathbb{R}$; signed measures; BV function, total variation, absolutely continuous, jump discontinuity; mutually singular measures.

2. State ...

   the Borel-Cantelli lemma, the monotone class theorem, Carathéodory's extension theorem, the great convergence theorems; Egorov's theorem and Lusin's theorem; the theorems of Fubini and Tonelli; Kolmogorov's extension theorem; the change of variables formula, Hölder's inequality,the Hardy-Littlewood maximal function theorem, Lebesgue's differentiation theorem in $\mathbb{R}^d$, Vitali's covering lemma;the Lebesgue-Radon-Nikodym theorem on a general measure space, on $\mathbb{R}^d$, and on $\mathbb{R}^1$; the Fundamental Theorem of Calculus.

3. Give an example of ...

   (a) a closed set of positive measure that has no interior;

   (b) a L^1-Cauchy sequence of functions which does not converge pointwise anywhere;

   (c) a sequence of functions $\{f_n\}$ in $L^1(\mathbb{R})$ converging to zero pointwise a.e. but such that

   $$\lim \int_{\mathbb{R}} f_n(x)\, dx \neq 0\,;$$

   (d) a continuous function on $[0,1]$ which is not absolutely continuous.

4. True or False?

   (a) If a real-valued function on $(0,1)$ is differentiable, then it is measurable.

   (b) If f is a measurable function on $[0,1]$, then the set

   $$C = \{x \in [0,1] : f \text{ is continuous at } x\}$$

   is measurable.

   (c) Every set of finite Lebesgue measure can be partitioned into two subsets of equal measure.

   (d) A subset of full Lebesgue measure in $(0,1)$ is necessarily dense.

   (e) A nowhere dense subset of $(0,1)$ has measure zero.

   (f) If $E \subset \mathbb{R}$ has Lebesgue measure zero, then $G = \{(x,y) \in \mathbb{R}^2 : x - y \in E\}$ has two-dimensional Lebesgue measure zero.

5. Let f be a nonnegative measurable function on $\mathbb{R}^d$ with

   $$m\left(\{x : f(x) > \lambda\}\right) = \frac{1}{1+\lambda^2}\,.$$

   Compute the L^1-norm of f. For what values of p is $f \in L^p$?

6. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $[0,1]$ with

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x)\, dm(x) < \infty\,.$$

Show that except for x in a set of measure zero, $f_n(x) \geq 1$ occurs only for finitely many n.

7. Let $\{f_n\}_{n\geq 1}$ and f be real-valued measurable functions on $\mathbb{R}$.

(a) If $f_n \to f$ a.e., show that for any positive number $\varepsilon > 0$,

$$\lim_{n\to\infty} m\left(\{x : |f(x) - f_n(x)| > \varepsilon\}\right) = 0\,.$$

(b) What can you say about the converse?

8. If $\{f_n\}$ is a *fast* Cauchy sequence in $L^1(\mathbb{R}^d)$, in the sense that $||f_n - f_{n-1}||_{L^1} \leq 2^{-n}$, prove that $\lim f_n(x)$ exists for almost every x.

9. For $c > 1$, find

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-cx}\, dx\,.$$

10. Let $f(\lambda, x)$ be a continuous function of two variables on the unit square $0 < \lambda, x < 1$. Suppose that the partial derivative $\frac{\partial f}{\partial \lambda}(\lambda, x)$ exists for all λ and x, and that

$$h(x) = \sup_{0<\lambda<1} \left|\frac{\partial f}{\partial \lambda}(\lambda, x)\right|$$

is integrable. Show that the function $F(\lambda) = \int_0^1 f(\lambda, x)\, dx$ is differentiable and satisfies

$$F'(\lambda) = \int_0^1 \frac{\partial f}{\partial \lambda}(\lambda, x)\, dx\,.$$

11. Let A be a positive definite, symmetric $d \times d$ matrix. Compute the Gaussian integrals

$$\int_{\mathbb{R}^d} e^{-x\cdot Ax}\, dm(x) \qquad \text{and} \qquad \int_{\mathbb{R}^d} |x|^2 e^{-x\cdot Ax}\, dm(x)\,.$$

12. If $f_n \to f$ in L^p and $g_n \to g$ in L^q, where $\frac{1}{p} + \frac{1}{q} = 1$, show that $f_n g_n \to fg$ in L^1.

13. (a) Assume that g is integrable on $(-1,1)$, and that

$$\int_{-1}^1 x^n g(x)\, dx = 0$$

for $n = 0, 1, 2, \ldots$. Prove that $g = 0$ a.e.

(b) Is it true that

$$\lim_{n\to\infty} \int_{-1}^1 x^n f(x)\, dx = 0$$

for every integrable function f?

(c) Let $\{e_n\}_{n\geq 0}$ be the sequence of Legendre polynomials, obtained by Gram-Schmidt orthonormalization from the sequence $\{x^n\}_{n\geq 0}$ in $L^2[-1,1]$. Given $f \in L^2[-1,1]$, show that there exist constants $\{a_n\}_{n\geq 0}$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x)$$

for a.e. x. How are the constants a_n determined? In what sense does the sequence converge?

14. Let α be an increasing function on $[0,1]$ such that $\alpha'(x)$ exists a.e. on $[0,1]$. Prove directly that

$$\alpha(1) - \alpha(0) \geq \int_0^1 \alpha'(x)\, dx\,.$$

Give an example where the inequality is strict.

15. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0, & x < 1, \\ (x-1)^2 + 1, & 1 \leq x < 2, \\ 2, & x \geq 2\,. \end{cases}$$

Define a Borel measure on $\mathbb{R}$ by $\mu((-\infty, x]) = F(x)$. Find the Lebesgue decomposition $\mu = \mu_a + \mu_c$ of μ, where μ_a is absolutely continuous with respect to Lebesgue measure, and μ_s is singular.

16. Let $f = \sum a_i \mathcal{X}_I$ be a step function, where $\{I_i\}_{i=1,\dots,N}$ is a disjoint collection of subintervals of $[0,1]$. Writing $I_i = (a_i, b_i]$, let

$$\ell(f) = \sum_{i=1}^{N} \left(\beta_i^{\frac{3}{4}} - \alpha_i^{\frac{3}{4}} \right)\,.$$

Show that

$$|\ell(f)|^2 \leq 2 \int_0^1 |f(x)|^2\, dx\,.$$

17. Let f and g be bounded nonnegative measurable functions on $(0,1)$. For $y > 0$, set

$$\phi(y) = \int_{E_y} f(x)\, dx\,, \quad \text{where} \quad E_y = \{x \in (0,1) : g(x) \geq y\}\,.$$

Prove that ϕ is decreasing and

$$\int_0^{\infty} \phi(y)\, dy = \int_0^1 f(x) g(x)\, dx\,.$$

18. Let

$$f(x) = x^{\alpha} \sin x^{-1}\,, \quad f(0) = 0\,.$$

For what values of $\alpha > 0$ is f of bounded variation on $[0,1]$?

19. If $r > 1$, for what values of $p \in [1, \infty]$ does $x^{-\frac{1}{r}}(1 + |\log x|)^{-1}$ belong to $L^p(0, \infty)$?

20. Let $f_n(x)$ be a sequence of increasing functions on $[0, 1]$ that converges pointwise to a function f. Prove that if f is continuous, then f_n converges *uniformly*.

21. Assume that $f \in L^p[0, \infty)$ for some $p \in [1, \infty)$, and that f is uniformly continuous on $[0, \infty)$. Prove that

$$\lim_{x \to \infty} f(x) = 0 .$$

22. Let $\{a_k\}$ be a sequence such that the series $\sum a_k x_k$ converges whenever $\sum |x_k|^2 < \infty$. Show that $\sum |a_k|^2 < \infty$.

23. (a) Show that $\frac{\sin x}{(1+x)^2}$ is integrable on $(0, \infty)$ but $\frac{\cos x}{1+x}$ is not.

    (b) Prove that

$$\lim_{R \to \infty} \int_0^R \frac{\cos x}{1 + x} \, dx = \int_0^\infty \frac{\sin x}{(1 + x)^2} \, dx .$$

24. Let f be a integrable function on $[0, 1]$, and consider $S = \{x \in [0, 1] : f(x) \text{ is an integer}\}$. Evaluate

$$\lim_{n \to \infty} \int_0^1 |\cos \pi f(x)|^n \, dx .$$

25. Let $f(x, t)$ be a function on the unit square such that $\partial_t f(x, t)$ exists for all x, t and satisfies

$$|f(x, t_1) - f(x, t_2)| \leq g(x)|t_1 - t_2|$$

    with some integrable function g. Prove that $f(\cdot, t)$ is integrable for each $t \in (0, 1)$, that $\partial_t(\cdot, t)$ is integrable for each t, and that

$$\frac{d}{dt} \int_0^1 f(x, t) \, dx = \int_0^1 \partial_t f(x, t) \, dx .$$

26. Give examples of sequences of measurable functions $\{f_n\}$, $\{g_n\}$ such that

    (a) f_n converges to zero a.e., but not in L^1;

    (b) g_n converges to zero in L^1, but $g_n(x)$ converges for no x.

27. Give examples of two increasing function F and G such that $F'(x) = G'(x) = 0$ a.e., F has discontinuities on a countable set x_i and $\sum\big(F(x_i^+) - F(x_i^-) = 1$ while G is continuous.

28. For which positive real number p does the integral

$$\int_0^1 \int_0^1 \frac{1}{(x^2 + y^2)^p} \, dx dy$$

    converge?

29. Let A be a Borel subset of the unit square $[0,1] \times [0,1]$. For $t \in [0,1]$, define the vertical and horizontal cross sections of A at t by

$$A_t = \{y \in [0,1] : (t,y) \in A\}, \quad A^t = \{x \in [0,1] : (x,t) \in A\}.$$

(a) Show that $A_{\frac{1}{2}}$ is a Borel set.

(b) If A_x is countable for each $x \in [0,1]$, show that $m(A^y) = 0$ for almost all y.

30. Let $Q = [0,1] \times [0,1]$ be the unit square. Show that

$$\int_Q \frac{1}{1-xy}\, dm = \sum_{n \geq 1} \frac{1}{n^2}.$$

31. Let f be an integrable function such that

$$\int_0^\infty x\,|f(x)|\, dx < \infty.$$

Prove that

$$\frac{d}{dt} \int_0^\infty \sin(xt)\, f(x)\, dx = \int_0^\infty x \cos(xt)\, f(x)\, dx.$$

32. If f is an integrable function on $\mathbb{R}^d$ such that

$$\int_E f(x)\, dx = 0$$

for every measurable set E, prove that $f = 0$ a.e.

33. Is it true that

(a) $\displaystyle\lim_{n\to\infty} \int_0^{2\pi} \frac{e^{inx}}{1+x}\, dx = 0.$

(b) $\displaystyle\lim_{\varepsilon\to 0^+} \int_0^\infty \frac{e^{-x}}{1+\varepsilon^2 x}\, dx = 1.$

34. Let f be a 2π-periodic continuously differentiable function. Use the Cauchy criterion to prove that its Fourier series converges uniformly. Argue that the limit is f.

35. Prove that

$$\lim_{n\to\infty} n \int_0^\infty e^{-x^2}(e^{\frac{x}{n}} - 1)\, dx = \int_0^\infty x e^{-x^2}\, dx = \frac{1}{2}.$$

36. Let f be an integrable function on $[0,1]$. Show that

$$\int_0^1 |s-t|^{-\frac{1}{2}} f(t)\, dm(t)$$

exists and is finite for almost every $s \in \mathbb{R}$.

37. Let f be absolutely continuous with derivative $f' \in L^p(\mathbb{R})$ for some $p > 1$. Find constants L, α such that

$$|f(x) - f(y)| \leq L|x - y|^{\alpha}, \quad x, y \in \mathbb{R}.$$

38. Evaluate

$$\lim_{n\to\infty} \int_{\mathbb{R}} (1 - e^{-\frac{t^2}{n}}) e^{-|t|} \cos t \, dt.$$

39. Let μ and ν be two finite Borel measures on the half-fine $(0, \infty)$. Show that there exists a unique finite measure ω which satisfies

$$\int_0^\infty f(z)\, d\omega(z) = \int_0^\infty \left\{ \int_0^\infty f(st)\, d\mu(t) \right\} d\nu(s).$$

Find an explicit formula for ω.

40. Assume that $E \subset \mathbb{R}$ has Lebesgue measure zero. Can the set

$$G = \{(x, y) \in \mathbb{R}^2 : x - y \in E\}$$

have positive Lebesgue measure?

41. Let f be a measurable function on $[0, \infty)$, and define

$$F(s) = \int_0^\infty \frac{f(x)}{(1 + sx)^2}\, dx.$$

(a) If $\frac{f(x)}{x}$ is integrable prove that $F(s)$ is finite a.e., and that F is integrable over $[0, \infty)$.

(b) If $f(x) \geq 0$ and $F(s)$ is bounded, then f itself must be integrable.

(c) Assume that f is continuous, and that $a := \lim_{x\to\infty} f(x)$ exists. Find

$$\lim_{s\to 0} sF(s), \quad \lim_{s\to\infty} sF(s).$$

42. Let $f(x, t)$ be a real-valued function on $\mathbb{R}^2$ such that $f(\cdot, t)$ is continuous for every $t \in \mathbb{R}$. Suppose there exists an integrable function g such that

$$|f(x, t)| \leq g(t), \quad \text{for all } x, t \in \mathbb{R}.$$

Prove that

$$F(x) = \int_{\mathbb{R}} f(x, t) \cos t \, dt$$

is bounded and continuous.

43. If $f \in L^1(0, 1)$ and $\int_0^x f(t)\, dt = x$ for every $x \in (0, 1)$, prove that $f(x) = 1$ a.e. on $(0, 1)$.

44. A function $f : (0, \infty) \to \mathbb{R}$ is *improperly Riemann integrable* if the Riemann integral

$$I(t) = \int_0^t f(x)\, dx$$

exists for every $t > 0$ and converges to some finite limit I as $t \to \infty$. If both f and $|f|$ are improperly Riemann integrable, prove that f is Lebesgue integrable and

$$I = \int_{(0,\infty)} f\, dm.$$