Theorem (Limits of measures are measures). *Let $(\mu_n)_{n\geq 1}$ be a sequence of measures on a measurable space $(X, \mathcal{M})$. Suppose that the limit*

$$\mu(A) := \lim_{n\to\infty} \mu_n(A) < \infty$$

exists for each $A \in \mathcal{M}$. Then μ is a measure.

Remark. This is sometimes called the Vitali-Hahn-Saks theorem. It has many applications in probability. Among its implications are strong compactness properties for spaces of probability measures. A generalization (to projection-valued measures) is useful in the analysis of self-adjoint operators on Hilbert spaces (through the Spectral Mapping Theorem).

Proof. The function μ is clearly nonnegative and finitely additive on $\mathcal{M}$ as a limit of the functions μ_n. In particular, μ is monotone under inclusions.

To show that μ is also σ-additive, let $(A_i)_{i\geq 1}$ be a sequence of disjoint sets in $\mathcal{M}$. It follows from the monotonicity and finite additivity of μ that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \;\geq\; \sup_n \mu\left(\bigcup_{i=1}^{n} A_i\right) \;=\; \sup_n \sum_{i=1}^{n} \mu(A_i) \;=\; \sum_{i=1}^{\infty} \mu(A_i)\,.$$

Using once more that μ is finitely additive, we write the difference between the leftmost and rightmost expressions as $L = \mu\left(\bigcup_{i\geq k} A_i\right) - \sum_{i\geq k} \mu(A_i)$, where $k \geq 1$ is arbitrary. Since the first term in this difference is nonincreasing in k by the monotonicity of μ, and the second term is the tail of a convergent series, we can take $k \to \infty$ to obtain

$$L \;=\; \lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} A_i\right)\,.$$

We will take advantage of the fact that $\mu_n(A)$ converges to $\mu(A)$ for *every* $A \in \mathcal{M}$ to show that L cannot be strictly positive. We construct a sequence of sets (B_n) and and a sequence of measures (ν_n) with the property that $\nu_m(B_k)$ is close to close to 0 for all $k > m$ and close to $\mu(B_k)$ for all $k \leq m$. Explicitly, we require that

$$|\nu_m(B_k)| \leq \frac{1}{m} \text{ for } k > m\,, \qquad |\nu_m(B_k) - \mu(B_k)| \leq \frac{1}{m\cdot 2^k} \text{ for } k \leq m\,. \tag{1}$$

In particular, $\lim \nu_n(B_n) = \lim \mu(B_n) = L$.

We define

$$B_n = \bigcup_{i=j_n}^{\infty} A_i\,, \qquad \nu_n = \mu_{\ell_n} \tag{2}$$

for a pair of increasing sequences of integers (j_n) and (ℓ_n) that we now specify recursively. For $n = 1$ we set $j_1 = 1$. Note that the first condition in Eq. (1) is empty for $k = 1$. Since $\lim \mu_n(B_1) = \mu(B_1)$, we can choose ℓ_1 large enough such that $|\mu_\ell(B_1) - \mu(B_1)| \leq 2^{-1}$ for $\ell \geq \ell_1$, and set $\nu_1 = \mu_{\ell_1}$. The the second condition in Eq. (1) is satisfied for $k, m \leq 1$.

Suppose we have already constructed $j_1, \ldots, j_{n-1}$ and $\ell_1, \ldots, \ell_{n-1}$ such that Eq. (1) is satisfied for $k, m < n$. Since ν_{n-1} is continuous from above, we can find an integer $j_n > j_{n-1}$ such that $\nu_{n-1}(B_n) \leq \frac{1}{n}$, where B_n is given by Eq. (2). Using that $B_k \subset B_n$ for all $k \geq n$ and the inductive assumption, we see that the first condition in Eq. (1) is satisfied for all $m < k = n$. Since the measures (μ_n) converge to μ, we can choose $\ell_n > \ell_{n-1}$ such that $|\mu_{\ell_n}(B_k) - \mu(B_k)| \leq 1/(n \cdot 2^k)$ for each $j = 1, \ldots, n$ and all $\ell \geq \ell_n$. Then the second inequality in Eq. (1) is satisfied also for $k \leq m = n$.

Finally, consider the set

$$C = \bigcup_{k=1}^{\infty} B_{2k} \setminus B_{2k+1} .$$

We estimate with the (reverse) triangle inequality

$$\begin{aligned} |\nu_{2n}(C) - \nu_{2n-1}(C)| &= \left| \sum_{k=1}^{\infty} \Big(\nu_{2n}(B_{2k} \setminus B_{2k+1}) - \nu_{2n-1}(B_{2k} \setminus B_{2k+1}) \Big) \right| \\ &\geq \big| \nu_{2n}(B_{2n} \setminus B_{2n+1}) - \nu_{2n-1}(B_{2n} \setminus B_{2n+1}) \big| \\ &\qquad - \sum_{\ell < 2n} |\nu_{2n}(B_\ell) - \nu_{2n-1}(B_\ell)| \\ &\qquad - \nu_{2n}(B_{2n+2}) - \nu_{2n-1}(B_{2n+2}) \end{aligned}$$

Here, the first term on the right hand side of the inequality accounts for the summand with $k = n$. The second term represents the maximal possible difference between the values of the two measures on the ring generated by the sets B_ℓ with $\ell < 2n$. For the last pair of terms, we have used that the annuli $B_{2k} \setminus B_{2k+1}$ with $k > n$ are disjoint subsets of B_{2n+2}. It follows with Eq. (1) that

$$|\nu_{2n}(C) - \nu_{2n-1}(C)| \geq \mu(B_{2n}) - \frac{4}{n} .$$

We finally take $n \to \infty$. The left hand side converges to zero, because $\nu_n(C)$ converges to $\mu(C)$, and the right hand side converges to L. Therefore $L = 0$.