Solutions to Assignment 1

January 31, 2021

1. A natural approach to this problem is to apply the chain rule a few times to the both sides of this ODE. For example, we will get

$$x''(t) = f'(x(t))x'(t) = f'(x(t))f(x(t))$$
$$x'''(t) = (f'(x(t))f(x(t)))' = f''(x(t))f(x(t))^2 + f'(x(t))^2 f(x(t)).$$

As you are dealing with second and third derivatives, you expect to obtain some convexity info about solutions. (try $f(x) = x^3$? the notion of an inflection point might be relevant)

2. The characteristic polynomial of this matrix is

$$0 = \lambda^2 - tr(A)\lambda + \det(A) = \lambda^2 - 2a\lambda + a^2 - bc = (\lambda - a)^2 - bc.$$

Therefore, the roots are $\lambda_{12} = a \pm \sqrt{bc}$. In other words, A has two distinct real roots. Finally. we can find the eigenvectors, which turn out to be $v_{12} = \begin{pmatrix} \pm\sqrt{\frac{b}{c}} \\ 1 \end{pmatrix}$. Therefore, our general solution looks like this:

$$X(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

3. In all of these case we want to find a solution via undetermined coefficients.

(a) As the hint suggests, try $x(t) = at^2 + bt + c$. Then our equation turns into

$$2at + b = (2a+1)t^2 + 2bt + 2c \Leftrightarrow (2a+1)t^2 + (2b - 2a)t + 2c - b = 0.$$

Therefore, $2a + 1 = 0, a = b, 2c = b$. A quick computation shows that

$$a = b = -\frac{1}{2}, c = -\frac{1}{4}.$$

(b) Once again, we just try plugging in $x(t) = Ce^{3t}$. We obtain

$$3Ce^{3t} = (2C+1)e^{3t} \Leftrightarrow (C-1)e^{3t} = 0.$$

Because e^x is never zero, we immediately get $C = 1$.

(c) Now it makes sense to try $x(t) = C_1 e^{3t} + C_2 te^{3t}$. By plugging this into the ODE, we get

$$3C_1 e^{3t} + C_2 e^{3t} + 3C_2 te^{3t} = 2C_1 e^{3t} + (2C_2 + 1)te^{3t}.$$

If we collect all terms, we get

$$(C_1 + C_2)e^{3t} + (C_2 - 1)te^{3t} = 0.$$

Using the fact that e^{at} and te^{at} are linearly independent (why?), we reduce our problem to solving the system

$$\begin{cases} C_1 + C_2 = 0 \\ C_2 - 1 = 0. \end{cases}$$

In the end, we obtain $C_2 = 1, C_1 = -1$.

(d) When dealing with sines and cosines, one might want to try a linear combination $x(t) = C_1 \cos(t) + C_2 \sin(t)$. By plugging this, we get

$$C_2 \cos(t) - C_1 \sin(t) = (2C_1 + 1) \cos(t) + 2C_2 \sin(t),$$

which is equivalent to

$$(2C_1 - C_2 + 1) \cos(t) + (2C_2 + C_1) \sin(t) = 0.$$

Knowing that $\cos(t)$ and $\sin(t)$ are linearly independent (why?) we get $C_1 = \frac{2}{5}, C_2 = -\frac{1}{5}$.

(e) It might seem like this part can be done exactly like (b), but 2 being in the exponent and in the ODE itself is no coincidence, plug in $x(t) := Ce^{2t}$ and you'll get

$$2Ce^{2t} = (2C + 1)e^{2t},$$

and we get $0 = 1$, which doesn't make sense! In such a case the only thing that one can do is try $x(t) = C_1 e^{2t} + C_2 te^{2t}$, you will end up with a nice system which has a solution.

p.s. Suppose going from e^{at} to a linear combination of e^{at} and te^{at} didn't work out either! (if you think about why this might happen, you can easily come up with an ODE where this is precisely the case) How to deal with this?

4. (a) It seems that we can just say that $x'(t) = f(x(t)) > 0$, and we are done. But we need to rule out the cases when $x(t) = 0$ or $x(t) = 1$. But this is precisely why we need f to be a C_1-function...

(b) It is natural to assume that an asymptote of $x(t)$ should be an equilibrium state, but we already know that there are none except $x(t) = 0$ or $x(t) = 1$. How to prove such a claim? First of all, because $x(t)$ is increasing and bounded by 1 sue to the existence-uniqueness, then $\lim\limits_{t\to\infty} x(t) \in [0, 1]$. If $x(t) \neq 0$, then $\lim\limits_{t\to\infty} x(t) > 0$.

Now recall that the Mean Value theorem tells us that for every $a > 0$ we have

$$x(a + 1) - x(a) = x'(c) = f(x(c(a)))$$

for some $c(a) \in (a, a + 1)$. Now, take $\lim\limits_{a\to\infty}$ of both sides:

$$0 = \lim_{a\to\infty} x(a + 1) - \lim_{a\to\infty} x(a) = \lim_{a\to\infty} x(a + 1) - x(a) = \lim_{a\to\infty} f(x(c(a))).$$

However, f is continuous and $\lim_{a\to\infty} c(a) = \infty$, so

$$\lim_{a\to\infty} f(x(c(a))) = f(\lim_{t\to\infty} x(t)).$$

Finally, we get that $f(\lim_{t\to\infty} x(t)) = 0$, but this can only be true for $\lim_{t\to\infty} x(t) = 1$. The argument for $\lim_{t\to-\infty} x(t)$ is very similar.

(c) The superposition principle implies that $y - x$ is a solution which satisfies the conditions of (a), and we are done.