MAT 267: Ordinary DifferentialEquations
Problem Set 2, due February 22, 2021

1. Chapter 4 # 3. (The phase portraits show the solution curves in the x-x'-plane. For the calculations, you can either use the corresponding first-order planar system, see Section 2.2, or else work directly in the second-order equation.)

2. Chapter 6 # 5

3. Consider the autonomous ODE

$$x' = f(x)\,, \tag{1}$$

where $x = (x_1, \ldots, x_n)$ and f is a vector field on $\mathbb{R}^n$. Let $x(t)$ be a solution, defined on some interval $(a, b) \subset \mathbb{R}$.

   (a) **Time translation.** Given $\tau \in \mathbb{R}$, show that the function $y(t) := x(t - \tau)$ is also a solution of Eq. (1). On which interval is $y(t)$ defined?

   (b) **Time reversal.** Show that the function $z(t) := x(-t)$ is a solution of the ODE

$$x' = -f(x)\,.$$

   On which interval is $z(t)$ defined?

   (c) **Time rescaling.** Given $\sigma > 0$, derive an ODE for the function $w(t) := x(\sigma t)$. On which interval is $w(t)$ defined?

4. **Euler-Cauchy equations** (the general theory behind Problem 3 on the midterm)
Consider the n-th order ODE

$$\sum_{k=0}^{n} a_k t^k x^{(k)} = 0\,, \quad t > 0\,. \tag{2}$$

Here, $a_0, \ldots, a_n \in \mathbb{R}$ are constants with $a_n \neq 0$.

   (a) Show that $x(t) = t^\gamma$ is a solution of this ODE, if and only if γ is a root of a certain polynomial,

$$P(\gamma) = \sum_{k=0}^{n} b_k \gamma^k\,.$$

   Find a formula for the coefficients $b_0, \ldots, b_n$.

It turns out that the Euler-Cauchy differential operator can be written as

$$\sum_{k=0}^{n} a_k t^k \left(\tfrac{d}{dt}\right)^k = \sum_{k=0}^{n} b_k \left(t\tfrac{d}{dt}\right)^k = P\left(t\tfrac{d}{dt}\right)\,. \tag{3}$$

(Since this is a little tedious, I am not asking you to verify it). Let's factor the polynomial as

$$P(\gamma) = \prod_{j=1}^{\ell} (\gamma - \gamma_j)^{m_j} ,$$

where $\gamma_1, \ldots, \gamma_\ell$ are the roots of P, and $m_1, \ldots, m_\ell$ are their multiplicities.

(b) Let Q be a polyomial of degree m, and $\gamma \in \mathbb{C}$. Prove that $y(t) := t^\gamma \, Q(\log t)$ satisfies

$$ty' - \alpha y = t^\gamma \, \tilde{Q}(\log t) ,$$

where $\tilde{Q}$ is a polynomial of degree $m - 1$ if $\alpha = \gamma$, and m otherwise.

(c) Conclude that any function of the form

$$x(t) = \sum_{j=1}^{\ell} t^{\gamma_j} \, Q_j(\log t) ,$$

where each Q_j is a polynomial of degree $m_j - 1$, solves Eq. (2).

(This is in fact the general solution but you are not asked to prove that. We have ignored the distinction between real and complex roots; as usual, non-real solutions come in conjugate pairs whose real and imaginary parts are real-valued solutions.)

Alternative solution of Problem 4. The change of variables $y(s) := x(e^s)$ (that is, $t = e^s$, $t\frac{d}{dt} = \frac{d}{ds}$ transforms Eq. (2) into the **constant-coefficient** equation

$$\sum_{k=0}^{n} b_k \left(\tfrac{d}{ds}\right)^k y = 0 , \tag{4}$$

see Eq. (3).

Example. In the special case

$$t^2 x'' - 3tx' + 4x = 0 ,$$

the polynomial factors as $P(\gamma) = \gamma(\gamma - 1) - 3\gamma + 4 = (\gamma - 2)^2$. We easily verify that

$$t^2 x'' - 3tx - +4x = \left(t\tfrac{d}{dt} - 2\right)^2 x ,$$

and obtain for the general solution

$$x(t) = t^2 (C_1 + C_2 \log t) ,$$

where C_1, C_2 are abitrary constants.