MAT 267: Ordinary DifferentialEquations
Problem Set 3, due March 10, 2021

1. Chapter 6 # 8, 9. (Try to give a verbal description of the dynamis, using words like *periodic, oscillation, growth/decay, stable/unstable, source/sink*. Emphasize how the two systems differ.)

2. Chapter 17 # 10. (Use the general solution of the ODE, then try to match the boundary conditions).

3. Explain the Inverse Function Theorem (assumptions and conclusions) in a few lines, using the words *existence*, *uniqueness*, *continuous dependence*, and *well-posed*.

4. **Global solutions.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and $L > 0$ a constant. Suppose that f satisfies the global Lipschitz condition

$$|f(x) - f(y)| \leq L|x - y|\,, \qquad \text{for all } x, y \in \mathbb{R}^n\,.$$

Define the function space

$$\mathcal{C}_M := \left\{ y : [0, \infty) \to \mathbb{R}^n \text{ continuous}\,, \sup_{t \geq 0} e^{-Mt}|y(t)| < \infty \right\}\,,$$

with norm $\|y\|_M := \sup_{t\geq 0} e^{-Mt}|y(t)|$. (You may take for granted that C_M is a complete metric space, with distance $d(x, y) = \|x - y\|_M$).

Fix $v \in \mathbb{R}^n$, and define, for $f \in C_M$, a function $Uf : \mathbb{R}_+ \to \mathbb{R}^n$ by

$$(Uy)(t) := v + \int_0^t f(y(s)\,ds\,, \qquad t \geq 0\,.$$

(This is the Picard map we considered in class.)

   (a) Find a condition on M such that U defines a contraction on C_M.
   (*Answer:* $M > L$; the contraction constant is $q = L/M$.)

   (b) Use this to prove that the initial-value problem

   $$y' = f(y) \quad (t > 0)\,, \qquad y(0) = v$$

   has a unique solution y in C_M.

   (c) Conclude that $y(t)$ is defined for all $t \geq 0$. Moreover, $|y(t)| \leq Const.\, e^{Mt}$.

(In fact, it is not hard to show that $y \in C_L$ but you are not asked to do this.)