MAT 267: Ordinary Differential Equations
Problem Set 4

1. **Exact equations.** Consider a differential equation of the form

$$a(x,t)x' + b(x,t) = 0\,, \tag{1}$$

where a, b are given smooth functions. Such an equation is called *exact*, if

$$\begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix} = \nabla V(x,t) \tag{2}$$

for some smooth function V, defined on a domain $D \subset \mathbb{R}^2$. The function V is called a *potential* for Eq. (1).

It is customary to write Eq. (1) in the form $a(x,t)dx + b(x,t)dt = 0$. The equation is exact if the 1-form $\omega := adx + bdt$ is exact, i.e., if $\omega = dV$.

   (a) Suppose Eq. (1) is exact, with potential V, and let $x(t)$ be a continuously differentiable real-valued function defined on an interval I, such that $(x(t), t) \in D$ for all $t \in I$. Prove that $x(t)$ solves Eq. (1), if and only if $V(x(t), t)$ is constant on I.

   (b) *Existence of a potential.* Under what conditions on a, b is Eq. (1) exact? Please explain how to construct a potential.

   (c) Suppose Eq. (1) is exact, with potential V. Given a point $(x_0, t_0) \in D$, under what conditions does there exist a solution $x(t)$ of Eq. (1) with initial value $x(t_0) = x_0$? What determines its maximal interval of existence?

   (d) Verify that the differential equation

$$(2xt^2 + 2)x' + (2x^2t - 2) = 0$$

   is exact, and find the solution with initial value $x(0) = 1$.

2. **Coupled mass-spring systems.** Chapter 6, Problem 7

3. **The mass-spring system.** The second-order ODE

$$mx'' + rx' + kx = 0\,,$$

describes the motion of an elastic spring with mass $m > 0$, spring constant $k > 0$, and friction coefficient and $r \geq 0$, centered at its equilibrium position.

   (a) *Energy.* Define a function

$$E(x,v) := \frac{m}{2}v^2 + \frac{k}{2}x^2\,, \qquad x, v \in \mathbb{R}\,.$$

   Assuming $x(t)$ is a solution of the system, compute $\frac{d}{dt}E(x(t), x'(t))$.

(b) *Non-dimensionalization.* Set $\omega_0 := \sqrt{k/m}$. (What is the role of ω_0 at $r = 0$?)
Change the time variable by setting $s = t\omega_0$, so that $d/dt = \omega d/ds$ to transform the equation to

$$x'' + \rho x' + x = 0\,, \tag{3}$$

where $\rho := \frac{r}{\sqrt{mk}} \geq 0$.

(c) Find the general solution of Eq. (3) for $\rho \geq 0$. (There are four cases.)
For each of the four cases, sketch the phase portrait in the x-x' plane, and briefly describe the dynamics and stability in a few words.

Remark on non-dimensionalization. In this context, *'dimension'* refers to the physical units associated with variables and parameters. In the mass-spring system,

- the parameter m has units of mass (kg);
- the friction coefficient r has units of force per velocity (Newton (m/sec)$^{-1}$=kg /sec);
- the spring constant k has units of force per length (Newton/meter = kg/sec^2),
- the frequency ω_0 has units of inverse time (Hz=sec^{-1}).

The transformation in Part (b) combines the physical quantities m, r, and k into the single dimension-less parameter ρ.

Non-dimensionalization is usually the first step in analyzing a physical system. After completing the analysis, the transformation must be reversed to obtain predictions in terms of the original physical parameters. Important dimensionless parameters are the Reynolds number Re (in fluid dynamics), the fine-structure constant α (in particle physics), and of course, π (the ratio of circumference to diameter of a circle).

4. **Resonance in the mass-spring system with friction.** Consider the inhomogeneous equation

$$x'' + \rho x' + x = \cos \omega t\,. \tag{4}$$

Here, ρ is the (non-dimensionalized) friction coefficient from Problem 1. The purpose of this problem is to analyze the long-term response of the mass-spring system to the *periodic forcing* $\cos \omega t$. The term *resonance* refers to the phenomenon is that the system response is strongest, if ω is close to the proper frequency of the system (in this case, $\omega_0 = 1$).

(a) Find the general solution of this equation. (The answer depends on ρ and ω.)
Hint: Set $x := \mathrm{Re}\, z$, where z solves the complex equation $z'' + \rho z' + z = e^{i\omega t}$.

(b) Given $\rho > 0$, find the unique solution that does not decay as $t \to \infty$.
Writing $x(t) = \mathrm{Re}\,(Ce^{i\omega t})$, determine the complex constant C.

(c) *Amplitude.* Set $a(\rho, \omega) = |C|$. Prove that a decreases with ρ, and $\lim_{\rho\to\infty} a(\rho, \omega) = 0$. (Work with $|a(\rho, \omega)|^{-2}$.)

(d) *Resonance frequency.* Show that for each $\rho > 0$, the amplitude $a(\rho, \cdot)$ assumes its maximum at a unique point $\omega_\rho < 1$. Verify that $\omega_\rho = 0$ for $\rho \geq 2$, and

$$\lim_{\rho\to 0^+} \omega_\rho = 1\,, \qquad \lim_{\rho\to 0^+} a(\rho, \omega_\rho) = +\infty\,.$$

Remark. By expressing C in polar coordinates, we can write the long-term response as $x(t) = a\cos(\omega t + \tau)$. Here a is the amplitude defined above, and τ is called the *phase shift* of the response relative to the forcing. With a similar analysis as for the amplitude, we can study how the the phase shift depends on ρ and ω. (You are not asked to do this).