# MAT 267: Ordinary Differential Equations
# Problem Set 5

1. *Determinant and trace in non-autonomous linear systems.* Consider the matrix equation

$$M' = A(t)M\,, \tag{1}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is a given matrix that depends continuously on time, and the unknown function $M$ takes values in $\mathbb{R}^{n\times n}$. Prove that $m(t) = \det M$ satisfies the scalar ODE

$$m' = \big(\operatorname{trace} A(t)\big)\, m\,.$$

*Hint:* Use the Taylor expansion $M(t+s) = M(t) + sM'(t) + o(s)$, then apply Eq. (1).

2. *The divergence.* Given a smooth vector $f$ field on $\mathbb{R}^n$, let $\Phi_t$ be the dynamical system associated with the system $x' = f(x)$ (also called the **flow** generated by the vector field $f$).

Let $A \subset \mathbb{R}^n$ be a bounded open subset, and let $A_t := \Phi_t(A)$ be its image under $\Phi_t$. We want to compute a formula for $\frac{d}{dt}\mathrm{Vol}\,(A_t)$ in terms of the vector field. By change of variables, (using $y = \Phi_t(x)$, $dy = |\det D\Phi_t(x)|\, dx$), we have

$$\mathrm{Vol}\,(A_t) = \int_A |\det D\Phi_t(x)|\, dx\,.$$

(a) Prove that

$$\frac{d}{dt}\det D\Phi_t(x) = \operatorname{div} f\big|_{\Phi_t(x)} \det \Phi_t(x)\,.$$

Here, $\operatorname{div} f := \operatorname{trace} Df = \sum_{i=1}^n \partial_{x_i} f^i$ is the **divergence** of $f$.

*Hint:* Recall the variational equation for $D\Phi_t$ from Section 7.4 (p. 151, top), then apply Problem 1(a). You can appeal to the semigroup property and consider only $t = 0$.

(b) Conclude that $\det D\Phi_t(x) > 0$, i.e., $\Phi_t$ is orientation-preserving. Moreover

$$\frac{d}{dt}\mathrm{Vol}\,(A_t)\Big|_{t=0} = \int_A \operatorname{div} f(x)\, dx\,. \tag{2}$$

(Use the Change of Variables from above and differentiate freely under the integral.)

(c) **Liouville's theorem in Hamiltonian mechanics.** Let $V$ be a smooth function on $\mathbb{R}^n$. Show that the flow defined on $\mathbb{R}^n \times \mathbb{R}^n$ (the **phase space**) by the system

$$\begin{aligned} x' &= y \\ y' &= -\nabla V(x) \end{aligned}$$

is volume-preserving: For any open bounded $A \subset \mathbb{R}^n \times \mathbb{R}^n$,

$$\mathrm{Vol}(\Phi_t(A)) = \mathrm{Vol}(\Phi(A))\,, \qquad t \in \mathbb{R}\,.$$

3. **The Dulac criterion.** Let $f$ be a vector field on $\mathbb{R}^2$ with $\mathrm{div} f > 0$. You will prove that the system $x' = f(x)$ has no non-constant periodic solutions.

   Suppose, for contradiction, that $x(t)$ is a non-constant periodic solution. By the Jordan curve theorem, the orbit of $x(t)$ separates $\mathbb{R}^2$ into two components, one of which is bounded and homeomorphic to a disk. Let $A$ be that bounded component.

   (a) Show that the flow $\Phi_t$ generated by the system maps $A$ diffeomorphically onto itself.
   (b) Use Problem 2(b) to show that $\mathrm{Vol}(\Phi_t(A))$ is strictly increasing.
   (c) Conclude!

4. **Grönwall's inequality.** Let $x(t)$ be a nonnegative continuous function on an interval $[0, T]$. Suppose that $x$ satisfies the integral inequality

$$x(t) \leq a + b \int_0^t x(s)\, ds\,, \tag{3}$$

   where $a$, $b$ are nonnegative numbers. You will prove that then $x(t) \leq ae^{bt}$ for $0 \leq t \leq T$. Note that we are not assuming that $x$ is differentiable!

   Let $C_T$ be the space of real-valued continuous functions on $[0, T]$, with the sup-norm. Define the Picard map $U : C_T \to C_T$ by

$$\big(U(x)\big)(t) := a + b \int_0^t x(s)\, ds\,, \qquad 0 \leq t \leq T\,.$$

   Write $x \leq y$ if $x(t) \leq y(t)$ for all $0 \leq t \leq T$.

   (a) *The Picard iteration is monotone.* Given a function $x(t) \in C_T$ that satisfies Eq. (3), set $x_0 := x$, and recursively $x_{k+1} = Ux_k$. Prove that the sequence of functions $(x_k)_{k \geq 0}$ increases with $k$, that is, $x_k \leq x_{k+1}$ for all $k \geq 0$.
   (b) *Comparison.* Set $T_1 = \min\{T, (b+1)^{-1}\}$, and let $y$ be the solution of the initial-value problem

$$y' = by\,, \qquad y(0) = a\,.$$

   Prove that $x_k(t) \uparrow y(t)$ for $0 \leq t \leq T_1$, and hence $x(t) \leq x_k(t) \leq ae^{bt}$ for $0 \leq t \leq T_1$.
   (c) *Extension.* If $T > T_1$, repeat the argument to obtain that $x(t) \leq ae^{bt}$ for all $0 \leq t \leq T$. (A small amount of book-keeping will be needed.)

**Generalized Grönwall's inequality.** (You are not asked to do this part.) The same strategy works if $x(t)$ is integrable (not necessarily continuous), and satisfies instead of Eq. (3)

$$x(t) \leq A(t) + \int_0^t b(s)x(s)\, dx\,, \tag{4}$$

where $A(t)$ is continuously differentiable on $[0, T]$, and $b(t) \geq 0$ is nonnegative and continuous. In that case, the conclusion is

$$x(t) \leq A(t) + \int_0^t A(s)b(s)e^{B(t)-B(s)}\, ds\,, \tag{5}$$

where $B(t) := \int_0^t b(s)\, ds$. The right hand side of Eq. (5) is the solution of $y' = A'(t) + b(t)y$ with initial value $y(0) = A(0)$.