MAT 267 Ordinary Differential Equations
Tutorial 2, January 22, 2020 (Section: Almut, 10am)

Matrix exponentials

For any square matrix $A \in \mathbb{R}^{n \times n}$, we define the matrix exponential function by the power series

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \,, \quad t \in \mathbb{R} \,.$$

We will see later in the course that this converges for all $t \in \mathbb{R}$, and yields a smooth (matrix-valued) function of t that satisfies $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$. In fact, $E(t) = e^{tA}$ it is the unique solution of the matrix-valued ODE $E' = AE$ with initial value $E(0) = I$. It is also called the **fundamental solution** of the ODE $x' = Ax$.

Let's compute . . .

1. . . . e^{tD}, where $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *(diagonal)*

2. . . . e^{tL} and e^{tR}, where $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *(nilpotent)*

3. . . . e^{tJ}, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *(skew)*

4. . . . e^{tA}, where $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ is the matrix discussed in class.
   (*Hint:* Diagonalize A. Its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.)

Further properties:

5. *Eigenvalues and eigenvectors*
   Suppose that v is an eigenvector of A, with eigenvlaue λ. Show that v is also an eigenvector of e^{tA}. What is the corresponding eigenvalue?

6. *Semigroup property*
   For any $s, t \in \mathbb{R}$, $e^{(s+t)A} = e^{sA} e^{tA}$; in particular, $e^{0A} = I$, and $e^{-tA} = (e^{tA})^{-1}$.

7. *Commuting matrices*
   If $AB = BA$ then $e^{t(A+B)} = e^{tA} e^{tB}$.

   This fails in general: For L, R as in (2.), $e^{t(L+R)}$, $e^{tL} e^{tR}$, and $e^{tR} e^{tL}$ are all different.