Def: $A \in \operatorname{Mat}(\mathbb{R}^n)$

matrix expon.

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \qquad t \in \mathbb{R}.$$

There is a diff between showing that

$$e^A v = \sum_{k=0}^{\infty} \frac{A^k v}{k!}$$

ptwse

$$\sum_{k=0}^{n} \frac{A^k}{k!} \to e^A \quad \text{uniformly!}$$

The fact that e^A is well-def for all matr. A is not trivial

$$\|Av\|_2 \leq C\|v\|_2$$

bold operator

$$\sup_{\|v\|_2 = 1} \|Av\|_2 = \|A\|$$

operator norm

$$\left\| \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\|_2 = \sqrt{v_1^2 + \dots + v_n^2}$$

$$\frac{d}{dt} e^{tA} = A e^{tA}$$ (you can prove this yourselves using chain rule)

1. $$e^{t\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = e^{\begin{pmatrix} ta & 0 \\ 0 & tb \end{pmatrix}} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{(ta)^k}{k!} & 0 \\ 0 & \frac{(tb)^k}{k!} \end{pmatrix} = \begin{pmatrix} \sum \frac{(ta)^k}{k!} & 0 \\ 0 & \sum \frac{(tb)^k}{k!} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{ta} & 0 \\ 0 & e^{tb} \end{pmatrix}$$

2. $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $$e^{tL} = e^{\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}} = E + tL + \frac{(tL)^2}{2} + \dots = E + tL = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$L^2 = 0$

Moral: If A is nilpotent, then e^{tA} is just a matrix polynomial in A

Motivation:

Suppose $X' = AX$ $\iff$ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$

$t \mapsto e^{tA}$ $t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ (vector-function)

Fundamental solution.

$X' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} X$ 2 and 3 are eigenvalues

$t \mapsto e^{t\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}$

$\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$ is a sol-n!

$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

eigenvalues

$$\underset{\sim}{A} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

$\lambda_1 = 3 \qquad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Expon. respects the changes of bases

$$\rightarrow C^{-1} \underset{\sim}{A} \underset{\sim}{C} = \underset{\sim}{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$e^{tC^{-1}AC} = \sum_{k=0}^{\infty} \frac{C^{-1}(tA)^k C}{k!} = C^{-1}\left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \right) C = C^{-1} e^{tA} C$$

$$e^{tC^{-1}AC} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = C^{-1} e^{tA} C \iff e^{tA} = \underset{\text{transition matrix}}{C} \underbrace{\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}}_{\text{know from eigenv}} C^{-1}$$

$$Av = \lambda v$$

$$e^{tA} v = \left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \right) v = \sum_{k=0}^{\infty} \frac{\overset{\overset{t^k\lambda^k v}{\parallel}}{(tA)^k v}}{k!} = e^{t\lambda} v$$

Prove this!

$$\boxed{\det(e^A) = e^{\mathrm{tr} A}}$$

7. $AB = BA \Rightarrow e^{A+B} = e^A e^B$

$$e^{A+B} \overset{1}{=} \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} \overset{2}{=} \sum_{k=0}^{\infty} \frac{\binom{n}{k} A^k B^{n-k}}{k!} \overset{3}{=} \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \left(\sum_{l=0}^{\infty} \frac{B^l}{l!} \right)$$

In general, $(A+B)^k = \sum_{l_1+\dots+l_m=k} A^{l_1} B^{l_2} A^{l_3} \dots A^{l_m}$

non-diag

$$e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \cdot e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

sA commutes with tA for $s, t \in \mathbb{R}$.

Suppose A is 2×2 mat.

When does $B \in Mat(\mathbb{R}^2)$ exist : $\exp(B) = A$

$\exp: Mat(\mathbb{R}^n) \to GL(\mathbb{R}^n)$ is not surjective

inv

1) Log

$$\mathrm{Log}(E+A) = \sum_{k=1}^{\infty} (-1)^k \frac{A^k}{k}$$

Check the sign

$\|A\| \leq 1$

what norm to choose?

2) Let $\underbrace{A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$ $= -E$

$$e^B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Conjecture: ?

A should be diag.

$\lambda_1, \ldots \lambda_n > 0$

Weird: $\exp: Mat(\mathbb{C}^n) \to GL(\mathbb{C}^n)$ is surjective. ?