MAT 267 Ordinary Differential Equations
Tutorial 3, January 29, 2020 (Section: Almut, 10am)

Consider the **constant-coefficient** (scalar, homogeneous linear) ODE of order n,

$$\sum_{k=0}^{n} a_k x^{(k)} = 0\,. \tag{1}$$

Here, $a_0, \ldots, a_n \in \mathbb{R}$ are constants, with $a_n \neq 0$, and $x^{(k)}(t)$ denotes the k-th derivative. We will see that this equation is much easier to solve than the corresponding first-order system:

1. Show that the **superposition principle** holds for (1), i.e., its solutions form a vector space.
2. For what values of λ is $x(t) = e^{\lambda t}$ a solution?
   (The answer will depend on the **characteristic polynomial** of the ODE).
3. Suppose that the characteristic polynomial has a root $\lambda = \alpha + i\beta$ with imaginary part $\beta \neq 0$. Find two linearly independent real-valued solutions of the ODE.
4. As an application, consider the **harmonic oscillator** equation

   $$mx'' + rx' + kx = 0\,,$$

   where $m > 0$ is the **mass**, $k > 0$ the **spring constant**, and $r \geq 0$ the **friction coefficient**. Take $m = k = 1$ (to simplify notation).

   (a) Depending on the value of r, what types of solutions do you find?
   Is the zero solution always stable?

   (b) What happens at $r = 2$? Find a basis for the solution space in this case.
   (Make an educated guess, or apply Variation of Constants.)

Construction of the general solution (...if you have time ...)

5. Let $y(t) = Q(t)e^{\lambda t}$, where $Q(t)$ is a polynomial of degree k and λ is a constant. If μ is any constant, prove that

   $$y' - \mu y = \tilde{Q}(t)e^{\lambda t}$$

   for some polynomial $\tilde{Q}$. What is its degree?
6. Find n linearly independent solutions of Eq. (1). (Factor the characteristic polynomial as $P(\lambda) = \prod_{i=1}^{\ell} (\lambda - \lambda_i)^{m_i}$ where $\lambda_1, \ldots, \lambda_\ell$ are the roots, and $m_1, \ldots, m_\ell$ their multiplicities, write the ODE as $\prod_{i=1}^{\ell} \left(\frac{d}{dt} - \lambda_i\right)^{m_i} x = 0$, and apply Part 5.)
7. Argue that the dimension of the solution space is n.
   (What initial values are needed to uniquely determine the solution?)

You have constructed a basis for the solution space!