Office hours:
a few just before midterms (also, MLC hours, Wed, 12-2 pm)

83644390156 (zoom)

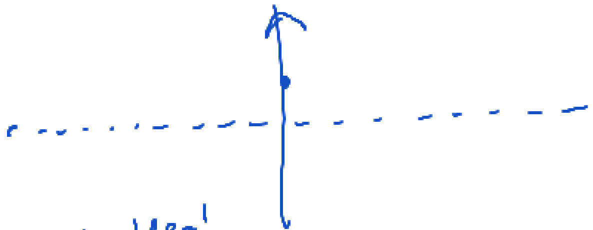
Def: A constant-coeff. ODE of order n is

(1) $\sum_{k=0}^{n} a_k x^{(k)} = 0.$ $\quad a_i \in \mathbb{R}$, $a_n \neq 0$.

$$x^{(k)} = \frac{d^k}{dt^k} x(t) = x^{\overbrace{''\cdots'}^{k \text{ times}}}(t)$$

0. a simple sol-n?

$x(t) :\equiv 0$ ← works!



E+U ⟹ a solution is either 0, or it never crosses x-axis! ✓

Rmk: this is not an initial value problem!

1. Suppose that $x(t)$, $y(t)$ are sol-ns to (1). $\begin{pmatrix} x \neq 0 \\ y \neq 0 \end{pmatrix}$

$$\sum_{k=0}^{n} a_k \left(\lambda x(t) + y(t)\right)^{(k)} \underset{\substack{\text{der} \\ \text{are} \\ \text{linear}}}{=} \sum_{k=0}^{n} a_k \left(\lambda x'(t) + y'(t)\right)^{(k-1)} = \sum_{k=0}^{n} a_k \left(\lambda x^{(k)}(t) + y^{(k)}(t)\right) = 0$$

$\lambda \in \mathbb{R}$

Moral: we can add two sol-ns
we can multiply by a constant.
(not a case for all ODE's!)

$$(e^{\lambda t})' = \lambda e^{\lambda t}$$
(for real λ)

2. $$\sum_{k=0}^{n} a_k \left(\frac{d^k}{dt^k} e^{\lambda t}\right) = \sum_{k=0}^{n} \lambda^k a_k e^{\lambda t} = \underbrace{e^{\lambda t}}_{\text{never zero!}} \underbrace{\left(\sum_{k=0}^{n} a_k \lambda^k\right)}_{\text{has to be } 0} \stackrel{?}{=} 0$$

Moral: $e^{\lambda t}$ is a sol-n $\Longleftarrow$ λ is a real root of $a_0 + a_1 x + \dots + a_n x^n$.
this is not enough, not all roots are real in general!

Ex: $x^2 + 1 = 0$ $\quad (x_{12} = \pm i)$

3. $\lambda = \alpha + i\beta$, $\beta \neq 0$

$$\sum_{k=0}^{n} a_k \left(\frac{d^k}{dt^k} \sin(\beta t)\right) = a_0 \sin(\beta t) + \beta a_1 \cos(\beta t) - \beta^2 a_2 \sin(\beta t) - \beta^3 a_3 \cos(\beta t) + \dots$$

$$\sum_{k=0}^{n} a_k \left(\frac{d^k}{dt^k} \cos(\beta t)\right) = a_0 \cos(\beta t) - \beta a_1 \sin(\beta t) - \beta^2 a_2 \cos(\beta t) + \dots$$

comp

$e^{\lambda t} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cdot e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$ ← this fn-n is not real-valued.

$a_0 + a_1\lambda + \dots + a_n\lambda^n = 0$

$\underbrace{e^{\alpha t}\cos(\beta t)}$ $+ i e^{\alpha t}\sin(\beta t)$

(try to do without $\mathbb{C}$)

$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}\left(e^{\lambda t}\cos(\beta t)\right)$ can be shown to be zero

we diff. in $\mathbb{C}$ to show that $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$ for $\lambda \in \mathbb{C}$

Suppose that

$\frac{d}{dt} f(t) = \frac{d}{dt}\left(\mathrm{Re}(f(t)) + i\,\mathrm{Im}(f(t))\right) =$

$= \mathrm{Re}\left(\frac{d}{dt} f(t)\right) + i\cdot \mathrm{Im}\left(\frac{d}{dt} f(t)\right)$

$a_0 + \dots + a_n\lambda^n = 0 \Rightarrow a_0 + a_1\bar{\lambda} + \dots + a_n\bar{\lambda}^n = 0$

$(+ i\sin(-\beta t))$

$e^{\bar{\lambda} t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$

$e^{\lambda t}, e^{\bar{\lambda} t}$ complex sol-ns

$e^{\bar{\lambda} t} = \overline{e^{\lambda t}}$

$\frac{e^{\lambda t} + e^{\bar{\lambda} t}}{2} = \mathrm{Re}$

Moral: λ is a root $\Rightarrow$ $\Rightarrow \bar{\lambda}$ is a root, $e^{\lambda t}, e^{\bar{\lambda} t}$ are complex sol-ns

$$f_1 = \frac{e^{\lambda t} + e^{\bar{\lambda} t}}{2} = e^{\lambda t}\cos(Bt)$$

$$f_2 = \frac{e^{\lambda t} - e^{\bar{\lambda} t}}{2i} = e^{\lambda t}\sin(Bt)$$

two lin indep sol-ns

$$f_1(0) = 1 \cdot \cos(0) = 1$$

$$f_2(0) = 1 \cdot \sin(0) = 0$$

not sol-ns to the same in. value problem

4. $mx'' + rx' + kx = 0$, $m = k = 1$.

$m\lambda^2 + r\lambda + k = 0$

$$\lambda_{12} = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

$r = 2$

$r > 2 \rightsquigarrow e^{\lambda_1 t}, e^{\lambda_2 t}$ $\quad C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$r < 2 \rightsquigarrow C_1 e^{\alpha t}\cos(Bt) + C_2 e^{\alpha t}\sin(Bt)$

$r = 2$ $\quad e^{2t} \leftarrow$ sol-n!

$$m(te^{2t})'' + 2(te^{2t})' + k te^{2t} = 0$$ exer!

Intuitively: $\sum a_k x^k = (x-\lambda_1)\dots(x-\lambda_n)$

$\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

you expect n-dim. space of sol-n,

$e^{\lambda_1 t}, \dots e^{\lambda_n t}$

$\uparrow$

lin indep.

Prove that

$e^{\lambda_1 t}, \dots e^{\lambda_n t}$ are lin.

$\frac{d}{dt} : C^\infty \to C^\infty$.

If $c_1 e^{\lambda_1 t} + \dots c_n e^{\lambda_n t} = 0$

take a derivative n times

Vandermonde determinant