$A \in \mathrm{Mat}_2(\mathbb{R})$ $\qquad \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

If $A, B \in \mathrm{Mat}_2(\mathbb{R})$, $[A,B] = 0 = AB - BA$. $\Rightarrow \exp(A+B) = \exp(A)\exp(B)$

$\forall t \in \mathbb{R} \quad \exp(t(A+B)) = \exp(tA)\exp(tB)$

$k \rightsquigarrow m$
$n \rightsquigarrow n+m$

Sketch:

$$\exp(A+B) = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}}{n!} \overset{=}{\sim} \sum_{\substack{0 \le n \le \infty \\ 0 \le k \le n}} \frac{\binom{n}{k}}{n!} A^k B^{n-k} \overset{=}{\sim}$$

$$\overset{=}{\sim} \underbrace{\left(\sum_{m=0}^{\infty} \frac{A^m}{m!} \right)}_{\exp A} \underbrace{\left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right)}_{\exp B} = \sum_{\substack{0 \le m \le \infty \\ 0 \le n \le \infty}} \frac{A^m B^n}{m!\,n!} = \sum_{\substack{0 \le m \le \infty \\ 0 \le n \le \infty}} \frac{(m+n)!}{\underline{m!\,n!}} \cdot \frac{A^m B^n}{(m+n)!} =$$

$$= \sum_{\substack{0 \le m \le \infty \\ 0 \le n \le \infty}} \underline{\frac{\binom{m+n}{m}}{(m+n)!} A^m B^n}.$$

$$\exp(A+B) = \sum_{\substack{0\le k\le n\\ 0\le n\le\infty}} \frac{\binom{n}{k}}{n!} A^k B^{n-k} \underset{\substack{k=m\\ n\to m+n}}{\rightsquigarrow} \sum_{\substack{0\le m\le\infty\\ 0\le n\le\infty}} \frac{\binom{m+n}{m}}{(m+n)!} A^m B^n =$$

$$\| $$

$$\sum_{\substack{0\le k\le n\\ 0\le n\le\infty}} \frac{A^k B^{n-k}}{k!(n-k)!}$$

$$= \sum_{\substack{0\le m\le\infty\\ 0\le n\le\infty}} \frac{A^m B^n}{m!\,n!} \overset{\text{def}}{=\!=} \left(\left(\sum \frac{A^m}{m!}\right)\left(\sum \frac{B^n}{n!}\right)\right)$$

Apply to: $x' = x$

$f: \mathbb{R}^2 \to \mathbb{R}$.

init value problem

$$\left\{\begin{array}{l} x' = f(x,t) \\ x(t_0) = x_0. \end{array}\right. \quad (1)$$

Thm: If f is a C^1-function in an open nbhd $\mathcal{U} = (t_0-\delta, t_0+\delta) \times (x_0-\varepsilon, x_0+\varepsilon)$, then there exists a unique sol-n to the initial value problem (1).

Thm: If f is C^1 w.r.t. the first coordinate, cont. w.r.t. the second coord.

(in other words, $\frac{\partial f}{\partial x}$ to be cont. in $\mathcal{U}$
f to be cont. in $\mathcal{U}$.)

savonger!

"Counterex": $x' = x^{1/3}$
$x(0) = 0$

$$\begin{cases} x' = x^2 \\ x(0) \neq 0 \end{cases}$$

$$\frac{x'}{x^2} = 1 \Rightarrow \int \frac{dx}{x^2} = t + C$$

$$\begin{cases} x' = x^2 \\ x(1) = 1 \end{cases} \Rightarrow C = 0.$$

$$\begin{cases} x' = x \\ x(0) = 1 \end{cases} \Rightarrow x(t) = e^t$$

$$-\frac{1}{x} = t + C$$

Typical question

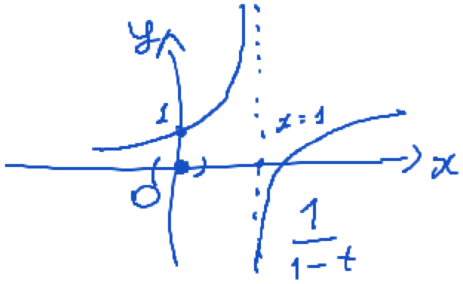
$$x = -\frac{1}{t+C} = \frac{1}{C-t}$$

$$\begin{cases} x'(t) = f(x) \\ x(t_0) = x_0 \end{cases} \Rightarrow \text{find maximal interval of existence}$$

For $C = 1 \Rightarrow x(t) = \frac{1}{1-t}$

$x(0) = 1$

"weird"



$x(t)$ is def'd on $(-\infty, 1)$

maximal interval of existence.

$$x' = 1 + x^2$$

$$\int \frac{dx}{1+x^2} = t + C = \arctan x$$

$$x(t) = \tan(t + C)$$

Exer: $x' = x^2$ blows up! $(\exists t_0: \lim_{t \to t_0} x(t) = \pm\infty)$, t_0 is finite)

$x' = x$ global, but fast sol-n

nice = x^a, e^{ax}, $\log(x)$

$$\begin{cases} x' = f(x) \\ x(0) = a \end{cases}$$

Suppose $f(x) = \Omega(\text{nice})$, $O(\text{nice})$ big-O, $o(\text{nice})$ small-o

if f is C^1, determine asymp. of $x(t)$

Why? $x' = \sin(x)$ can't be solved in elem f-ns.

local behavior $\rightsquigarrow$ linearize $\quad f(x) \rightsquigarrow f'(x_0)x$

$x' = f(x)$

$x' = f'(x)x$

global behaviour ?