MAT 267 Ordinary Differential Equations
Tutorial 5, February 12 (Section: Almut, 10am)

Vector fields and diffeomorphisms

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field with bounded derivative, $\sup_{x\in\mathbb{R}^n} ||Df(x)|| < \infty$. Consider the ODE

$$x' = f(x)\,, \tag{1}$$

where $x = (x_1, \ldots, x_n)$. After the break, we will establish **global existence and uniqueness** of solutions for such equations, i.e., for every $x_0 \in \mathbb{R}^n$ the ODE has a unique solution $x(t)$, defined for all $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, define a function $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_t(x_0) := \text{the solution of (1) with initial value } x(0) = x_0\text{, evaluated at time } t\,.$$

You may take for granted that Φ depends smoothly on t and x_0.

1. **Example (Twin Peaks gradient flow).** Consider the vector field on $\mathbb{R}^2$ given by

   $$f(x,y) = -\nabla V(x,y)\,, \qquad \text{where} \quad V(x,y) := 1 - \tfrac{1}{2}(x-1)^2(x+1)^2 + \tfrac{1}{2}y^2\,.$$

   Explicitly, the resulting system of ODE is $x' = 2x(x-1)x+1)$, $y' = -y$.

   Use the contour plot for V to sketch the phase portrait for this system of ODE.
   What are the steady-states? How does $f = -\nabla V$ relate to the level sets?

2. Describe the maps Φ_t in this example in terms of the graph of V.
   (It may be helpful to consider the cross section at $y = 0$).

Turning to the general problem in Equation (1):

3. Explain why $\Phi_0(x_0) = x_0$, i.e., Φ_0 is the identity map, and

   $$\frac{d}{dt}\Phi_t(x_0) = f(\Phi_t(x_0))\,, \qquad t \in \mathbb{R}\,.$$

   (No calculation required; a few well-placed words will do.)

4. **The semigroup property.** Prove that $\Phi_{s+t} = \Phi_s \circ \Phi_t$, i.e.,

   $$\Phi_{s+t}(x_0) = \Phi_s(\Phi_t((x_0))\,, \qquad s,t \in \mathbb{R}\,.$$

5. Conclude that for each $t \in \mathbb{R}$, the map $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection, with inverse Φ_{-t}.

Further questions:

6. In the special case where f is linear, what is the diffeomorphism Φ_t?
7. Conversely, suppose that $\{\Psi_t\}_{t\in\mathbb{R}}$ is a family of diffeomorphisms that depends smoothly on t and satisfies the semigroup property. Construct a vector field, g, such that

$$\frac{d}{dt}\Psi_t(x) = g(\Psi_t(x))\,, \qquad t \in \mathbb{R}, x \in \mathbb{R}^n\,.$$

   Hint: First consider $\frac{d}{dt}\Psi_t(x)\big|_{t=0}$.
8. In the special case where each Ψ_t is linear (in x), show that there is a matrix A such that

$$\Psi_t(x) = e^{tA}x\,, \qquad t \in \mathbb{R}, x \in \mathbb{R}^n\,.$$