# MAT 267 Ordinary Differential Equations
# Tutorial 6, February 26 (Section: Almut, 10am)

## Banach's Contraction Mapping Theorem

Let $d(x, y)$ be a metric on a set $X$. Assume that the metric space $X$ is complete.

1. Remind yourself what those terms mean. Which subsets of $\mathbb{R}^n$ are complete?

Let $F : X \to X$ be a function and $q$ a constant with $0 \leq q < 1$.

$$d(F(x), F(y)) \leq q d(x, y)\,, \quad \text{for all } x, y \in X\,.$$

Such a function is called a **contraction** on $X$. You will prove that $F$ has a unique **fixed point**, i.e., the equation $F(x) = x$ has exactly one solution in $X$.

2. **Uniqueness.** Prove that $F$ can have at most one fixed point.

3. **Existence.** Given a point $a \in X$, set $x_0 := a$ and define recursively $x_{n+1} = F(x_n)$, $n \geq 0$.

   (a) Find a constant $C$ (depending on $a$) such that $d(x_n, x_{n+1}) \leq Cq^n$ for all $n \geq 0$.
   (b) Show that the sequence $(x_n)$ converges. (Use the Cauchy criterion.)
   (c) Let $x^* := \lim x_n$. Provide an explicit upper bound on $d(x_n, x^*)$.
   (d) Show that $x^*$ is a fixed point of $F$.

You have proved the Contraction Mapping Theorem!

Note that the assumptions on $X$ are not restrictive (for instance, $X$ may be unbounded). Can the assumptions on $F$ be relaxed?

4. **Assumptions on $F$.** Suppose you only know that $d(F(x), F(y)) < d(x, y)$. What parts of the theorem continue to hold (uniqueness, or existence)? Support your conclusion with a sketch of the function $f(x) = \frac{x}{1+x}$ for $x \geq 0$.

We finally investigate how the fixed point depends on parameters. Suppose that $F$ depends on another parameter, $s$. Assume that $F(\cdot, s)$ is a **uniform contraction**,

$$d(F(x, s), F(y, s)) \leq q d(x, y)\,, \quad \text{for all } x, y \in X \text{ and all } s\,.$$

Let $x^*(s)$ be the unique fixed point of $F(\cdot, s)$, defined by $F(x^*(s), s)) = x^*$.

5. **Continuous dependence.** If $F(x, s)$ is continuous in $s$, prove that $x^*(s)$ is continuous.

   (Fix $s$. Given $\varepsilon > 0$, suppose that $t$ is such that

   $$d(F(x^*(s), s), F(x^*(s), t)) \leq \varepsilon\,.$$

   Establish the estimate

   $$d(x^*(s), x^*(t)) \leq \frac{\varepsilon}{1-q}\,.$$