MAT 267 Ordinary Differential Equations
Tutorial 7, Marh 5 2021 (Section: Almut, 10am)

Global Existence and Uniqueness

Let f be a vector field on $\mathbb{R}^n$. In class, we proved a theorem on **local Existence and Uniqueness** of solutions to the system

$$x' = f(x)\,. \tag{1}$$

1. Remind yourself what the existence and uniqueness theorem says about solutions of Eq. (1)? Remember to give assumptions as well as conclusions.

You will now construct **global** solutions of this problem.

2. **Global uniqueness.** Suppose the system has the following local uniqueness property:

   If two solutions $x_1(t)$ (defined on an interval I_1) and $x_2(t)$ (defined on I_2) satisfy $x_1(s) = x_(s)$ for *one* point $s \in I_1 \cap I_2$, then $x_1(t) = t_2(t)$ for *all* $t \in I_1 \cap I_2$.

   In that case, the two solutions can be combined to a solution $x(t)$ in the union $I_1 \cup I_2$.

   Conclude that any solution of Eq. (1) has a unique extension to a maximal time interval $I_{\max} = (T_{\min}, T_{\max}) \subset \mathbb{R}$.

3. **Globally Lipschitz vector fields.** Suppose for the moment that there is a constant L such that
$$|f(x) - f(y)| \leq L|x - y| \qquad \text{for all } x, y \in \mathbb{R}^n\,.$$
   Showe that for any initial value v, Eq. (1) has a (unique) solution, defined for $t \in \mathbb{R}$, with initial value $x(0) = v$.

   Hint: Iterate the local existence theorem to constuct the solution on intervals $[0, k\tau]$ for $k = 1, 2, \ldots$. (What determines the value of $\tau > 0$?)

4. **Blow-up condition.** In general (if f is not globally Lipschitz), fix a solution $x(t)$, and let $I_{\max} = (T_{\min}, T_{\max}) \subset \mathbb{R}$. be its maximal time of existence. Prove that

   - either $T_{\max} = +\infty$, that is, the solution exists for all > 0,
   - or $|x(t)| \to \infty$ as $t \uparrow T_{\max}$.

   Hint: Prove the contrapositive: For any $T > 0$,
$$R := \liminf_{t \to T} |x(t)| < \infty \qquad \Longrightarrow \qquad s < T_{\max}\,.$$
   (Argue that whenever $|x(s)| \leq R$ for some s, you can extend the solution by a fixed amount, τ, up to time $s + \tau$. Use that the ball $B := \{x \in \mathbb{R}^n : |x| \leq R\}$ is compact.)

So the only way a solution can cease to exist is by running off to infinity.

Further question: Consider solving Eq. (1) on a domain $D \subset \mathbb{R}^n$. Here,f is a smooth vector field on D, and we are looking for solutions $x(t)$ lying in D.

5. **Global existence on bounded domains.** Assume D is bounded. Prove that, for every $v \in D$, Eq. (1) has a unique solution $x(t)$ with initial value $x(0) = v$.

   Prove that the maximal time of existence satisfies

   - either $T_{\max} = +\infty$, that is, the solution exists and remains in D for all > 0,
   - or the solution reaches the boundary, that is, $\operatorname{dist}(x(t), \partial D) \to 0$ as $t \uparrow T_{\max}$.

6. **Global existence on manifolds.** Let M be a smooth compact manifold, and f a smooth vector field on M.

   Prove that for every point $p \in M$, Eq. (1) has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$. (Apply the previous result to coordinate charts.)

   Conclude that the vector field defines a group of diffeomorphims $\{\Phi_t\}_{t \geq 0}$ on M.