MAT 267 Ordinary Differential Equations
Tutorial 9, March 26, 2021 (Section: Almut, 10am)

Foxes and rabbits

Consider the **Lotka-Volterra system**

$$\begin{aligned} x' &= x(a - by) \\ y' &= y(-c + dx)\,, \end{aligned}$$

where a, b, c, d are positive parameters. This is one of the first models in polulation ecology, describing the interaction between two species: **predator** and **prey**.

1. Which of the two components x and y describes the predator, which describes the prey?

2. Find the **equilibria** of the system, and write down the matrix of the **linearization** about each of them. Classify the linearizations according to stability and type.

3. What can you conclude for the nonlinear system from the linear analysis in Part 2? Sketch two or three conceivable phase portraits that are compatible with the linear analysis.

Global dynamics. To understand how solutions of the Lotka-Volterra system really behave on the positive quadrant, we need more information. Note that if x and y are positive at one time, they remain positive for all time. (Why?)

4. **Nullclines** are curves with $x' \equiv 0$ or $y' \equiv 0$. (Obviously, they intersect at the equilibria.)

   Find the nullclines and use them to improve your sketches. Indicate the regions of the x-y-plane where $x' > 0$ ($x' < 0$) and $y' > 0$ ($y' < 0$).

5. **Lyapunov function.** Let

   $$L(x, y) := dx - c\log x + by - a\log y\,, \qquad x > 0, y > 0\,.$$

   Compute $\frac{d}{dt}L(x(t), y(t))$ and argue that L is constant along solutions.

6. Sketch a contour plot of L in the positive quadrant and use that to draw a phase portrait for the Lotka-Volterra system. (Note that L is strictly convex and assumes its minimum at the positive equilibrium point.)

   Conclude that **every positive, non-constant solution is periodic**.

The Lotka-Volterra system can be used to predict the impact of environmental factors (such as hunting) on the two populations. This works best in simple ecosystems that have only few other species — most famously for arctic foxes and rabbits.

You find a more thorough analysis of this model in Section 11.2 of Hirsch-Smale-Devaney.