APM 351: Differential Equations in Mathematical Physics
Test 1, November 25 2009

(Four problems; 20 points each.)

1. (a) Solve the initial-value problem

$$2u_x + 3xu_y = u\,, \quad u(0, y) = g(y)\,.$$

(b) Sketch a few characteristics. Does your solution exist on the entire plane $\mathbb{R}^2$? Is it unique? Why?

(c) Suppose we try to solve the same PDE with initial values $u(x, 0) = g(x)$, what goes wrong?

2. (a) Write down a solution of the diffusion equation

$$u_t = u_{xx}\,, \quad (x \in \mathbb{R}, t > 0)$$

with initial values $u(x, 0) = \phi(x)$.

(b) Assume that ϕ is a continuous function with $0 \leq \phi(x) \leq M$, and that ϕ vanishes outside some compact interval $[-R, R]$. Show that the solution from (a) satisfies $0 < u(x, t) < M$ for all $x \in \mathbb{R}$, $t > 0$.

(c) Does the diffusion equation have *finite speed of propagation*? Explain!

3. (a) Consider the Fourier series

$$\cos\frac{x}{2} = \sum_{n=-\infty}^{\infty} A_n e^{inx}\,, \quad (-\pi < x < \pi)\,.$$

Argue that that the Fourier coefficients are real, $A_{-n} = A_n$, and that $A_n = 0$ when n is odd.

(b) Determine the coefficients A_n.

(c) Briefly comment in which sense the series converges.

4. (a) Define what it means for a problem to be *well-posed*.

(b) Consider Poisson's equation

$$\Delta u(x) = f(x)\,, \quad (x \in U)$$

on a bounded connected open set $U \subset \mathbb{R}^n$ with smooth boundary. Impose inhomogeneous Dirichlet boundary conditions $u(x) = g(x)$ for $x \in \partial U$. Use the maximum principle to prove *uniqueness*, i.e., the problem has at most one solution.

(c) Give another proof of uniqueness for the Poisson problem from (b), using the energy method.