MAT 1060: Partial Differential Equations I
Assignment 2, September 23 2007

Read Chapter 2 up to p. 64, and Chapter 8 up to p. 435.

Please hand in to Ehsan in class on Wednesday, October 7:

- Chapter 2 (p. 85): Problems 5, 7, 8, 9, 10, 13. In Problem 5, you may find it useful to consider the functions
$$v_{\pm}(x) = u(x) \pm \frac{\max|f|}{2n}\big(|x|^2 - 1\big)\,.$$
- Chapter 8 (p. 487): Problem 3.

Additional problem:

- **The minimal surface equation:** Let u be a smooth real-valued function on a bounded open set U. The surface area of the graph of u is given by
$$\mathcal{S}(u) = \int_U (1 + |Du(x)|^2)^{1/2}\,dx\,.$$
(a) Assume that u minimizes $\mathcal{S}$ among all functions with given boundary values on U. Show that u satisfies the minimal surface equation
$$\sum_{i=1}^n \left(\frac{u_{x_i}}{(1+|Du|^2)^{1/2}}\right)_{x_i} = 0\,,$$
by considering variations $\mathcal{S}(u + t\phi)$ for smooth functions ϕ with compact support in U.

  (b) Verify that the minimal surface equation is quasilinear.

  (c) Show that that $\mathcal{S}$ is *convex* in u, i.e., if u, v are two functions on U and $0 < t < 1$, then
$$\mathcal{S}\big((1-t)u + tv\big) \le (1-t)\mathcal{S}(u) + t\mathcal{S}(v)\,.$$
Hint: Write the function $h(p) = \sqrt{1 + |p|^2}$ as the composition of two convex functions.

  (d) Show that smooth solutions of the minimal surface equation are uniquely determined by their boundary values: If $u, v \in C^2(U) \cap C(\overline{U})$ both solve the minimal surface equation on U, and $u = v$ on ∂U, then $u = v$ on U.
Hint: When can equality hold in (c)?