MAT 1060: Partial Differential Equations I
Assignment 3, October 6 2009

Read Chapter 2 to the end.

Please hand in to Ehsan in on Wednesday, October 21:

- Chapter 2 (p. 85): Problems 15, 16, 17, 18;

and the additional problems:

- *Lorentz invariance of the wave equation in* $\mathbb{R}^3$.

  For $z = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, consider the quadratic form $m(z) = |x|^2 - t^2$. A **Lorentz transformation** is an invertible linear transformation $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that satisfies

  $$m(L(z)) = m(z)\,, \quad (z \in \mathbb{R}^4)\,.$$

  Lorentz transformations form a group, i.e., the product and the inverse of Lorentz transformations is again a Lorentz transformation. (Why?)

  (a) Let Γ be the diagonal $(n+1)\times(n+1)$-matrix with $\Gamma_{ii} = 1$ for $i \leq n$ and $\Gamma_{n+1,n+1} = -1$. Show that L is a Lorentz transformation, if and only if its matrix satisfies $L^t\Gamma L = \Gamma$. What can you say about the determinant of L?

  (b) If L is a Lorentz transformation, and $v(z) = u(L(z))$, show that

  $$u_{tt} - \Delta u = v_{tt} - \Delta v\,.$$

  Hint: Let ϕ be a smooth function with compact support on $\mathbb{R}^n$, and consider the integral

  $$I(u) = \int\int \big(u_{tt} - \Delta u\big)\phi(x,t)\,dxdt\,.$$

  Integrate by parts and change variables.

  Remarks: (i) The quadratic form m is called the **Minkowski metric** on $\mathbb{R}^4$. It is used to define the geometry of space-time in the Theory of Special Relativity. A vector z is called **timelike** if $m(z) < 0$, **spacelike** if $m(z) > 0$, and **Null** (or **characteristic**) if $m(v) = 0$.

  (ii) The Lorentz transformations are the fundamental symmetries of Minkowski space, analogous to the orthogonal transformations on Euclidean space. An important difference is that the Lorentz group is non-compact, while the orthogonal group is compact. Can you see, why?

- *(Alternate proof of the maximum principle).*

  Let u be a smooth real-valued function on $\mathbb{R}^n$. Suppose that

  $$-\Delta u + b(x)\cdot Du \leq 0\,, \quad x \in U\,,$$

where U is a bounded set, and b is a bounded vector-valued function on U. Prove that u satisfies the weak maximum principle, i.e., it assumes its maximum on the boundary.
Hint: Consider the function

$$v(x) = u(x) + \varepsilon e^{\lambda x_1} ,$$

and choose $\lambda > 0$ so that $-\Delta v(x) + b(x) \cdot Dv(x) < 0$.

Remarks: (i) A careful analysis shows that u satisfies the *strict* maximum principle, see Chapter 6.4.2. The proof remains valid, if the Laplacian is replaced by a differential operator of the form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j u ,$$

where the functions a_{ij} are smooth and bounded, the matrix $A(x) = \big(a_{ij}(x)\big)_{i,j=1}^{n}$ is symmetric, and its eigenvalues lie in some interval $[c, C]$, where c and C are positive constants.

(ii) Maximum principles are particularly useful in nonlinear problems, where exact solutions are usually not available, but they are largely limited to elliptic and parabolic second-order equations for a single real-valued function.

(iii) References: The strong maximum principle is due to Hopf (1927). Good modern sources are "Maximum Principles in Differential Equations", by Protter and Weinberger (book; Springer 1967), and "The strong maximum principle revisited", by Pucci and Serrin (review article, J. Differential Eq. **196**:1-66, 2004).