# MAT 1060: Partial Differential Equations I
# Assignment 6, November 18 2009

Read Sections 5.2-5.7 of Evans. Start with the definition and the properties of the weak derivative in Section 5.2. In Sections 5.6 and 5.7, concentrate on the the statements of the Sobolev inequality on p. 263, Morrey's inquality on p. 266, and the Rellich-Kondrachev conpactness theorem on p. 272. (Omit the more technical results). We will discuss the main ideas of the proofs in class. As supplementary reading, you may enjoy Chapters 6 and 9 of Folland's *Real Analysis*, and Chapters 2, 6 and 8 of Lieb and Loss' *Analysis*. Lieb and Loss also have nice treatments of harmonic functions (Chapter 9) and Poisson's equation (Chapter 10).

Please hand in to Ehsan on Friday, December 4:

- Chapter 5 (p. 289): Problems 5, 6, 8, 10, 12, 17.

In Problem 17, I'd suggest to start with (ii), and then write

$$u = u^+ - u^- \,, \quad |u| = u^+ + u^-$$

to get (i) and (iii). Consider only the cases $1 < p < \infty$.

- Additional problem: The *Hardy-Littlewood-Sobolev* inequality says that for any two smooth functions of compact support, and any $0 < \lambda < n$,

  $$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} \, dxdy \le C||f||_{L^p}||g||_{L^p} \,,$$

  with a constant $C = C(n, \lambda)$ and an exponent $p = p(n, \lambda) \in [1, \infty]$. Use scaling to identify $p$.

  (b) Let $\Phi$ be the fundamental solution of Laplace's equation on $\mathbb{R}^3$. Use the Hardy-Littlewood-Sobolev inequality to show that the solution operator $f \mapsto \Phi * f$ defines a bounded linear transformation from $L^{6/5}$ to $L^6$.

  (c) Consider Poisson's equation

  $$-\Delta u = f$$

  on $\mathbb{R}^3$. If $f$ is smooth and compactly supported, show that the unique bounded solution of Poisson's equation satisfies

  $$\int_{\mathbb{R}^3} |Du|^2 \, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} \, dxdy \,.$$

  Conclude that

  $$||Du||^2_{L^2} \le \frac{C}{4\pi} ||f||^2_{L^{6/5}} \,.$$

  *Hint:* Consider $\int fu \, dx$.

  *Remark:* The double integral on the right hand side is called the *Coulomb energy* of the charge distribution $f$.