PARTIAL DIFFERENTIAL EQUATIONS — LECTURE 3

BURGER'S EQUATION: SHOCKS AND ENTROPY SOLUTIONS

A **conservation law** is a first order PDE of the form

$$u_t + \partial_x F(u) = 0\,.$$

We think of x as a spatial variable, and t as time. The function u is interpreted as the density of a (one-dimensional) fluid, and F describes the flow speed as a function of density. In gas dynamics, F is usually increasing and convex, while in traffic modeling it is increasing and concave. The simplest example is **Burger's equation**, where $F(u) = \frac{1}{2}u^2$, resulting in

$$u_t + uu_x = 0\,.$$

The results we discuss are representative for scalar conservation laws in one dimension. Systems of conservation laws in higher spatial dimensions, which appear in fluid dynamics, pose greater challenges that are beyond the scope of this course.

Conservation laws are examples of **quasilinear** equations, that is, equations that are linear in the highest order derivatives, with coefficients that depend on the unknown function. What makes them interesting is that singularities can develop after a finite time, even when the initial values are smooth. This motivates the

- **weak formulation** of the PDE: We ask that for every interval $[a, b]$,

$$\frac{d}{dt}\int_a^b u(x,t)\,dt + F(u(x,t))\Big|_{x=a}^{b} = 0\,.$$

The prototypical singularity of a weak solution is a **shock**, where the value of the solution jumps across a smooth curve $x = \gamma(t)$. The motion of the shock is governed by the

- **Rankine-Hugoniot condition**

$$\gamma'(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r}$$

  at every point $(\gamma(t), t)$ on the curve.

For Burger's equation, the shock speed is just the average of the characteristic speeds immediately to the left and right of the shock. We can think of the characteristic ODE as an infinitesimal version of the Rankine-Hugoniot condition.

It turns out that weak solutions at a given initial-value problem are not unique. Uniqueness is restores, by requiring additionally that the solution satisfy

- **Lax' entropy condition**: At a shock,

$$F_u(u_\ell) > \gamma'(t) > F_u(u_r)\,.$$

This means that nearby characteristics should always run into the shock. Characteristics emanating from a shock are viewed as unphysical. Note that the entropy condition breaks the symmetry of the PDE under the change of variables $(x,t) \to (-x,-t)$, and introduces a preferred direction of time. This is reminiscent of the second law of thermodynamics, which says that entropy always increases with time. One consequence is that shocks travel forward in gas dynamics, but backwards in traffic modeling. A weak solution that satifies Lax' condition is called an **entropy solution**.

We note in passing that the method of characteristics can be adapted to solve first order fully nonlinear equations locally, i.e., in a neighborhood of the initial data. Examples of fully nonlinear first order equations are the eikonal equation $|\nabla u| = 1$ (which describes characteristic surfaces for the wave equation), and the Hamilton-Jacoby equation $u_t + H(u, \nabla u) = 0$ (which appears in classical mechanics).

PROBLEMS

(1) Compute explicitly the solution of **Burger's equation** $u_t + uu_x = 0$ with initial values

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

that satisfies both the Rankine-Hugoniot jump condition and the entropy condition. Please draw a sketch of the characteristics and the shocks.

(2) Let u be an entropy solution of the conservation law

$$u_t + \partial_x F(u) = 0\,,$$

where F is a non-decreasing function on the real line with $F(0) = 0$. Justify the following statement: If F is convex, then shocks travel forward (to the right); if F is concave, they travel backwards (to the left).

(3) *(The real and imaginary parts of holomorphic functions are harmonic)*
  (a) Let u, v be two real-valued functions in two variables. Assume that (u, v) satisfy the Cauchy-Riemann system

$$u_x = v_y \quad u_y = -v_x\,.$$

  Show that u and v satisfy Laplace's equation
  (b) Conversely, assume that $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies Laplace's equation. (We say that u is a **harmonic function**). Show that there exists a function v such that the Cauchy-Riemann differential equations hold. (v is called the **conjugate harmonic function** to u.)
  Hint: The vector field $(-v_x, v_y)$ is *exact*.
  (c) What happens to (b) if u is defined not on the entire plane but on an open subset $\Omega \subset \mathbb{R}^2$?

(4) A **plane wave** is a solution of the wave equation $u_{tt} = \Delta u$ with $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$ of the form $u(x,t) = f(k \cdot x - ct)$, where f is a smooth function of a single variable, c is a constant, and k is a constant vector in $\mathbb{R}^3$. Find all the three-dimensional plane waves.

(5) Use the method of characteristics to solve
  (a) $xu_x + yu_y = 2u$, $u(x,1) = \phi(x)$;
  (b) $u_x + u_y = u^2$, $u(x,0) = \phi(x)$;
  (c) $uu_x + u_y = 1$, $u(x,x) = \frac{1}{2}x$.
  Which of these equations are linear?

(6) A **multi-index** is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers. Define
  - $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the **order** of α,
  - the power $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$,
  - the **factorial** $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_n!$ and the **multinomial coefficient** $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1! \cdot \cdots \cdot \alpha_n!}$.

  (a) Show that $(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$. (*Hint:* Induction over either k or n).
  (b) Let f be a k-times continuously differentiable function in n variables. Write down the Taylor expansion of order k about $x = 0$, using multi-index notation.