

PARTIAL DIFFERENTIAL EQUATIONS — DAY 2

THE TRANSPORT EQUATION: METHOD OF CHARACTERISTICS

The general **first order linear PDE** has the form

$$V(x) \cdot \nabla u + c(x)u = d(x),$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $V = (V_1, \dots, V_n)$ is a vector field. The **method of characteristics** reduces this PDE to a system of coupled ODEs as follows:

- Determine the **characteristics** in \mathbb{R}^n by solving $\frac{dx}{dt} = V(x)$.
- Along the characteristics, the PDE reduces to ODE $\frac{d}{dt}z + cz = d$, where $c = c(x(t))$, $d = d(x(t))$ are the values of the coefficients along the characteristic. The solution $z(t)$ gives the values of $u(x(t))$ along the characteristic curve. The curves $(x(t), z(t))$ in \mathbb{R}^{n+1} are called the **characteristic curves** of the PDE.
- Assign **initial values** for u along some hypersurface S , making sure that S intersects each characteristic exactly once. In practice, we parametrize the surface by $S = \{x_0(s)\}$, where the parameter s ranges over some subset of \mathbb{R}^{n-1} , and describe the values of u on the surface by a function $g(s)$.
- **Construction of the solution.** To find the value of u at a point x , determine the characteristic curve through x by solving the first system of ODE. Follow it to the point x_0 where it intersects the initial surface S . Parametrize the characteristic so that $x(0) = x_0 \in S$, let s_0 be the corresponding parameter value. Solve the remaining ODE for z with initial value $z(t_0) = g(s_0)$. The value of this solution at $s = 0$ is the desired value of $u(x) = z(0)$.

Note that the characteristic ODEs can be nonlinear, even when the PDE is linear, and hence its solutions may not be defined globally. Even if the characteristic equations have global solutions, the final step may be problematic. The Inverse Function Theorem guarantees that we can express (s, t) in terms of x , and thus solve for $u(x)$ in some neighborhood of the initial surface S , provided that it intersects the characteristics **transversally**, i.e.,

$$\det(V(x_0(s)), Dx_0(s)) \neq 0.$$

Here, the first column is tangent to the characteristic, and the remaining $n - 1$ columns span the tangent space of the initial surface S .

An important special case is the **transport equation**

$$u_t + V(x, t) \cdot \nabla u = f(x, t),$$

with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where the initial hypersurface is given by $S = \{(x, t) \in \mathbb{R}^{n+1} \mid t = 0\}$. Note that the transversality condition is automatically satisfied, since the t -axis is orthogonal to S . In that case, it is convenient to identify the parameter t in the method of characteristics with the time variable, and to parametrize S by $x_0(s) = s$. Transport equations play an important role in fluid dynamics and optimal transportation.

The method of characteristics directly applies to the **quasilinear equation**

$$V(x, u) \cdot \nabla u = c(x, u).$$

Here, the system for the characteristics (for $x(t)$) depends on the last ODE (for $z(t)$), and the transversality condition involves also the initial values g . We will see tomorrow that this can cause the formation of shocks and other nonlinear phenomena.

PROBLEMS

- (1) Let $\rho(x, t)$ describe the temperature distribution in a fluid (such as the North Atlantic). Assume that the velocity of the fluid is given by a known vector field, $V(x, t)$ (such as the Gulf Stream).
 - (a) Use the divergence theorem to derive a PDE for the temperature ρ that models the impact of the fluid. (This is the **continuity equation**.)
 - (b) Modify your PDE to account also for the diffusion of heat.
- (2) Consider the function $\phi(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ for $t > 0$ and $x \in \mathbb{R}$. Here, $k > 0$ is a constant. Verify that ϕ satisfies the heat equation $u_t = k u_{xx}$. What happens in the limits as $t \rightarrow 0$ and $t \rightarrow \infty$?
- (3) Solve $2u_x + 3u_y + u = e^x$ with $u(x, 0) = 0$.
- (4) Solve the initial-value problem for the transport equation

$$u_t + bu_x = f(x, t)$$

with initial values $u(x, 0) = g(x)$. Is the problem well-posed?

- (5) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. Please sketch some characteristics! In what region of the plane is the solution uniquely determined? If you enlarge the region, what fails — existence or uniqueness or both?
- (6) Consider the PDE

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad (r > 0, \theta \in \mathbb{R})$$

with periodic boundary conditions in θ , i.e., $u(r\theta + 2\pi) = u(r, \theta)$ for all θ .

(We will see later that this is Laplace's equation in polar coordinates.)

- (a) Set $u(r, \theta) = f(r)g(\theta)$ and separate variables to obtain a pair of ODE for f and g .
- (b) Solve these ODE to obtain special solutions for the PDE.
(Hint: Try $f(r) = r^\alpha$.)
- (7) Fix $\alpha \in \mathbb{R}$, and let $f : \mathbb{R}^n \setminus \{0\}$ be a smooth real-valued function.
 - (a) Assume that f is **homogeneous of degree** α , that is,

$$f(tx) = t^\alpha f(x) \quad \text{for all } t > 0, x \neq 0.$$

Show that f satisfies **Cauchy's differential equation** $x \cdot \nabla u = \alpha u$.

- (b) Sketch the characteristics of Cauchy's differential equation in the plane, and verify that the unit sphere $S = \{x \in \mathbb{R}^n : |x| = 1\}$ satisfies the transversality condition.
- (c) Conclude that every solutions of Cauchy's differential equation is homogeneous.
- (8) (*The semigroup of dilations on \mathbb{R}^n*)
Fix $\alpha, \beta \in \mathbb{R}$. Find the general solution of the transport problem

$$u_t + \alpha x \cdot \nabla u = 0$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.