PARTIAL DIFFERENTIAL EQUATIONS — DAY 1

WHAT IS A PDE? HOW CAN WE "SOLVE" IT?

A **partial differential equation** (PDE) is an equation of the form

$$F(x, u, Du, ..., D^k u) = 0\,, \tag{1}$$

where

- $u = u(x_1, \dots, x_n)$ is the unknown function, also called the **dependent variable**;
- the integer $k \geq 1$ is called the **order** of the PDE;
- $D^k u$ means all the k-th order partial derivatives of u; and
- the **independent variables** $x = (x_1, \dots, x_n)$ range over some open connected set $\Omega \subset \mathbb{R}^n$.

A **solution** of the PDE is a k-times continuously differentiable function $v(x_1, \dots, x_n)$ such that

$$F(x, v(x), Dv(x), ..., D^k v(x)) = 0 \qquad \text{for all } x \in \Omega\,.$$

Note that F and u may be vector-valued, in which case we call the PDE a **system**.

We will mostly consider first- and second order equations in two, three, or four variables. Ideally, we would like to represent the solution of a given PDE explicitly in terms of its boundary values. It turns out that this is possible only for a small number of classical equations. Among these are the transport equation $u_t + V(x) \cdot \nabla u = 0$, Poisson's equation $\Delta u = f$, the heat equation $u_t = \Delta u$, the wave equation $u_{tt} - \Delta u = 0$, and Schrödinger's equation $iu_t + \Delta u - W(x)u = 0$, all of which are fundamental in Physics. These equations are all **linear**, i.e., they can be written as

$$Lu = f\,,$$

where L is a linear differential operator and f is a given function. Nonlinear equations of order $k = 1$ are also well-understood.

For most other equations, there are no solution formulas. In fact, almost nothing can be said about a *general* non-linear PDE. What we know about important equations grows out of the special properties of the underlying physical problem, such as energy conservation, entropy inequalities, and symmetries. Each class of equations requires its own set of analytical techniques and its own solution theory. Thus, numerical and asymptotic methods play a huge role in the field.

Fundamental analytical questions are:

- Does there **exist** a solution for a given PDE?
- Under what additional boundary conditions is the solution **unique**?
- Does the solution **depend continuously on the data**?

A boundary-value problem that guarantees existence, uniqueness, and continuous dependence of solutions is called **well-posed**. Well-posedness is crucial, if one wants to evaluate the solution of a PDE numerically, because in an ill-posed problem, even small approximation errors can have a devastating effect. But there is no reason for a general PDE to be well-posed. There even is a famous example of a linear PDE that has no solutions!

Still, one can often find at least some solutions for a given PDE, using cheap tricks such as **Separation of Variables** to reduce it to an ordinary differential equation (ODE). Sometimes we can produce all solutions of the PDE from these special solutions, using the

- **Superposition principle:** If the PDE is linear and homogeneous, then any linear combination of solutions is again a solution.

If the coefficients of the linear PDE are constant, then we can translate and differentiate solutions to obtain other solutions; if the equation has additional symmetries (such as rotations and dilations), they can be used to generate yet more solutions.

PROBLEMS

(1) By trial and error, find a solution of the heat equation $u_t = u_{xx}$ with initial values $u(x,t) = x^2$.

(2) (a) In one sentence, explain the geometric meaning of the **divergence theorem**

$$\int_\Omega \nabla \cdot F(x)\, d^n x = \int_{\partial\Omega} F(x) \cdot n(x)\, dS(x)\,.$$

(b) If F is a smooth vector field on $\mathbb{R}^3$ and $|F(x)| \leq |x|^{-3}$, prove that

$$\int_{\mathbb{R}^3} \nabla \cdot F\, dx = 0\,.$$

Hint: Consider a large ball B_R and then take $R \to \infty$.

(3) Let $u(x,t)$ denote the temperature at time t, for x in a region $\Omega \in \mathbb{R}^n$. **Fourier's law** says that the heat flow is given by the vector field

$$F(x,t) = -c\nabla u(x,t)\,,$$

that is, heat flows from hot to cold, in proportion to the temperature gradient. Here, $c > 0$ is a constant that depends on the medium.

(a) Use the divergence theorem to derive a PDE for u.
(b) On physical grounds, what intital and boundary conditions would you add?
(c) How would you change the equation to model the heat flow in an inhomogeneous medium? Please write down the resulting PDE for u, and explain your choice.

(4) Check that each of the following equation have solutions of the form $u(x,y) = e^{\alpha x + \beta y}$. Find the possible values of the constants α, β for each example.

(a) $u_x + u_y + u = 0$;
(b) $u_{xx} + u_{yy} = 5e^{x-2y}$;
(c) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$.

(5) Use separation of variables to find solutions of the diffusion problem $u_t = ku_{xx}$ for $0 < x < \ell$ and $t \in \mathbb{R}$ that satisfy the mixed boundary conditions $u(0,t) = u_x(\ell,t) = 0$.

(6) Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and consider the PDE

$$u_t = p(u)u_x\,.$$

If a differentiable function u satisfies the relation $u(x,t) = f(x + t\,p(u))$, where f is a differentiable function, check that u is a solution of the PDE. Use this to construct some special solutions for the following equations:

(a) $u_t = ku_x$, where k is a constant;
(b) $u_t = uu_x$;
(c) $u_t = (u \sin u)u_x$.

In each of the above cases, does the superposition principle hold? Please explain!

(7) (a) Show that there exists a unique solution for the system

$$u_x = 3x^2 y + y$$
$$u_y = x^3 + x$$

together with the initial value $u(0,0) = 0$. What is the general solution of the PDE?

(b) What happens if you replace the coefficient '3' in the first line by '2.99'?