MAT 1060: PDE I

Final Exam
December 11, 2007; 2-5pm
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(5 problems, each worth 20 points).

1. Use the method of characteristics to solve the initial-value problem

$$\begin{cases} 2xtu_x + u_t = u\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = x\,, & x \in \mathbb{R}, t = 0\,. \end{cases}$$

2. Let u solve the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times t = 0\,, \end{cases}$$

where g, h are smooth functions with compact support.

(a) Show that the total energy

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3} u_t^2 + |Du|^2\,dx$$

is constant in time.

(b) Show that there exists a constant C (which depends on the initial conditions) such that

$$|u(x,t)| \le \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and all $t > 0$.

3. Let $U \subset \mathbb{R}^n$ a smooth bounded domain, and fix $\varepsilon > 0$. Define, for $u \in W^{1,p}$, the functional

$$\mathcal{I}(u) = \frac{1}{2}\int_0^T \int_U e^{-t/\varepsilon}\left(|Du|^2 + \varepsilon u_t^2\right)dxdt\,.$$

(a) Suppose that u is a smooth function on $U \times (0,T)$ that is a critical point for $\mathcal{I}$ (with the appropriate boundary conditions). Find the Euler-Lagrange equation for u by considering variations $\frac{d}{ds}\mathcal{I}(u+s\phi)\big|_{s=0}$ for suitable test functions ϕ on $U \times (0,T)$.

Remark: Formally setting $\varepsilon = 0$ yields the heat equation.

(b) Let u_ε be the solution of your PDE from (a) with initial and final conditions

$$u(x,0) = 0,\ u(x,T) = g(x), \quad \text{for } x \in U$$

and boundary conditions

$$u(x,t) = 0, \quad \text{for } x \in \partial U, t \in [0,T]\,.$$

Here, g is a smooth function with compact support in U. It is known that u_ε is smooth and satisfies the same maximum principle as solutions of Laplace's equation in $n+1$ variables (you are not asked to prove this). As $\varepsilon \to 0$, does u_ε converge to a solution of the heat equation? (Does such a solution even exist?) Discuss this question in view of the appropriate maximum principles.

4. (a) The *Hardy-Littlewood-Sobolev* inequality says that for any two smooth functions of compact support, and any $0 < \lambda < n$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda}\, dxdy \le C||f||_{L^p}||g||_{L^p}\,,$$

with a constant $C = C(n,\lambda)$ and an exponent $p = p(n,\lambda) \in [1,\infty]$. Use scaling to identify p.

(b) Let Φ be the fundamental solution of Laplace's equation on $\mathbb{R}^3$. Show that the solution operator $f \mapsto \Phi * f$ defines a bounded linear transformation from $L^{6/5}$ to L^6.

(c) Consider Poisson's equation

$$-\Delta u = f$$

on $\mathbb{R}^3$. If f is smooth and compactly supported, show that the unique bounded solution of Poisson's equation satisfies

$$\int_{\mathbb{R}^3} |Du|^2\, dx = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|}\, dxdy\,.$$

Conclude that

$$||Du||^2_{L^2} \le \frac{C}{4\pi}||f||^2_{L^{6/5}}\,.$$

Hint: Consider $\int fu\, dx$.

Remark: The double integral on the right hand side is called the *Coulomb energy* of the charge distribution f.

5. Define and explain the term *"finite speed of propagation"*, using the transport equation and the wave and heat equations in $\mathbb{R}^3$ to illustrate your definition.