MAT 1060: Partial Differential Equations I
Assignment 2, September 16 2007

Read Chapter 2 up to p. 53, and Chapter 8 up to p. 435.

Please hand in to Wenbin Kong's mailbox by noon on Friday, October 5:

- Chapter 2 (p. 85): Problems 3, 4, 5, 7, 8. In Problem 5, you may find it useful to consider the functions
$$v_\pm(x) = u(x) \pm \frac{\max |f|}{2n}\big(|x|^2 - 1\big)\,.$$

Additional problem:

- Let u be a smooth real-valued function on a bounded open set U. The surface of the graph of u is given by
$$\mathcal{S}(u) = \int_U (1 + |Du(x)|^2)^{1/2}\,dx\,.$$
(a) Assume that u minimizes $\mathcal{S}$ among all functions with given boundary values on U. Show that u satisfies the *minimal surface equation*
$$\sum_{i=1}^n \left(\frac{u_{x_i}}{(1 + |Du|^2)^{1/2}}\right)_{x_i}\,,$$
by considering variations $\mathcal{S}(u + t\phi)$ for smooth functions ϕ with compact support in U.

  (b) Verify that the minimal surface equation is quasilinear.

  (c) Show that $\mathcal{S}$ is *strictly convex* in u, i.e., if u, v are two different functions on U and $0 < t < 1$, then
$$\mathcal{S}\big((1-t)u + tv\big) < (1-t)\mathcal{S}(u) + t\mathcal{S}(v)\,.$$
Hint: Write the function $h(p) = \sqrt{1 + |p|^2}$ as the composition of two convex functions to obtain $\leq$, and then analyze the equality cases.

  (d) Show that smooth solutions of the minimal surface euqation are uniquely determined by their boundary values, i.e., if u, v are two smooth solutions and $u = v$ on ∂U, then $u = v$ on U.