MAT 1060: Partial Differential Equations I
Assignment 3, October 17 2007

Reading Chapter 2 to the end. Please hand in to Wenbin Kong's mailbox by noon on Friday, October 26:

- Chapter 2 (p. 85): Problems 9, 10, 13, 15,

including the additional problem:

- *(Alternate proof of the maximum principle)*

  Let u be a smooth real-valued function on $\mathbb{R}^n$.

  (a) Assume that u satisfies

  $$-\Delta u + b(x) \cdot Du < 0 \,, \quad x \in U \,,$$

  where U is a bounded set, and b is a bounded vector-valued function on U. Apply the first- and second-derivative test to see that u cannot have a local maximum in the interior of U.

  (b) Suppose u satisfies the weaker inequality

  $$-\Delta u + b(x) \cdot Du \leq 0 \,, \quad x \in U \,.$$

  Prove that u satisfies the weak maximum principle, i.e., it assumes its maximum on the boundary.
  Hint: Consider the function

  $$v(x) = u(x) + \varepsilon e^{\lambda x_1} \,,$$

  and choose $\lambda > 0$ so that $-\Delta v(x) + b(x) \cdot Dv(x) < 0$.

  (c) Let now w be a smooth real-valued function on $\mathbb{R}^n \times (0, \infty)$. Assume that w satisfies

  $$w_t - \Delta w + b(x) \cdot Dw \leq 0 \,, \quad (x,t) \in U \times (0,T] \,,$$

  where U is again a bounded open set in $\mathbb{R}^n$, and $0 < T < \infty$. Prove that U assumes its maximum on the parabolic boundary $\Gamma_T = \partial U \times (0,T] \cup U \times \{0\}$.

  Remarks: (i) A careful analysis shows that u and w satisfy *strict* maximum principles, see Chapter 6.4.2. The proof remains valid, if the Laplacian is replaced by a differential operator of the form

  $$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j u \,,$$

  where the functions a_{ij} are smooth and bounded, the matrix $A(x) = \big(a_{ij}(x)\big)_{i,j=1}^n$ is symmetric, and its eigenvalues lie in some interval $[c, C]$, where c and C are positive constants.

  (ii) Maximum principles are particularly useful in nonlinear problems, where exact solutions are usually not available, but they are largely limited to elliptic and parabolic second-order equations for a single real-valued function.

  (iii) References: The strong maximum principle is due to Hopf (1927). Good modern sources are "Maximum Principles in Differential Equations", by Protter and Weinberger (book; Springer 1967), and " The strong maximum principle revisited", by Pucci and Serrin (review article, J. Differential Eq. **196**:1-66, 2004).