APM 351: Differential Equations in Mathematical Physics
Assignment 10, due December 1, 2011

Summary

Poisson's equation

$$\Delta u = f$$

describes physical systems in equilibrium such as steady states for diffusion equations (with f describing sources and sinks), Maxwell's equation for a static electric field (with charge distribution given by f), and expected values of many interesting random variables under Brownian motion starting at x. It is the prototype of an **elliptic** equation.

The most important case is **Laplace's equation**

$$\Delta u = 0$$

whose solutions are called **harmonic** functions. In one dimension, the only harmonic functions are the linear, $u(x) = ax + b$. In two dimension, every harmonic function is the real part of a holomorphic function. As such, it is **analytic** (i.e., smooth, and agrees with its Taylor series), and satisfies the **strong maximum principle** (i.e., u cannot assume a local maximum or minimum on a connected domain unless it is constant). We will show that harmonic functions in higher dimensions share the last two properties. A useful fact is the **mean value property**, which says that the average of a harmonic function over a ball or a sphere agrees with its value at the center.

We will discuss methods for solving Poisson problems, consisting of Poisson's equation on a domain together with specified boundary values. The goal is to recover u from f and the given boundary data. In the special case of Laplace's equation on the unit disc, **Poisson's formula**

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \tilde{\theta}) + r^2} \, g(\tilde{\theta}) \, d\tilde{\theta}$$

provides the unique harmonic function u on the disc with boundary values $u(\cos\theta, \sin\theta) = g(\theta)$.

The **Poisson kernel**

$$P_r(\theta) = \frac{1}{2\pi} \cdot \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

has the properties that

- $P_r(\theta) > 0$ for all $0 \leq r < 1$ and all θ;

- $\int_0^{2\pi} P_r(\theta) \, d\theta = 1$ for all $0 \leq r < 1$;

- $\lim_{r \to 1} P_r(\theta) = 0$ for all $\theta \neq 0$.

Thus, Poisson's formula represents the solution as an average of the boundary data, weighted by the Poisson kernel P_r. Since the weight concentrates at $\theta = 0$ as $r \to 1$, it follows that $P_r \to \delta$, in the sense that

$$\lim_{r \to 1} \int_0^{2\pi} P_r(\theta) h(\theta) \, d\theta = h(0)$$

for every continuous 2π-periodic function h. By definition, the **delta distribution** is the linear transformation that sends the function h to its value at zero,

$$\delta(h) = \int_0^{2\pi} \delta(\theta)h(\theta)\,d\theta = h(0).$$

Remark. Poisson's formula is analogous to the integral representation of solutions to the heat equation on the real line in terms of the initial data (see Assignment 4, top of second page). The Poisson kernel plays the role of the fundamental solution, and the limit $r \to 0$ in Poisson's formula corresponds to the limit $t \to 0$ in the heat equation.

Assignments:

Read Chapter 6 of Strauss.

1. Find the sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$.

2. Let f be a continuously differentiable 2π-periodic function.

   (a) How are the Fourier coefficients a_n of f related to those of f'? Argue that $\sum_{n=\infty}^{\infty} n^2 |a_n|^2$ converges.

   (b) Consider the partial sums S_N and S'_N of the Fourier series of f and f', respectively. Apply Schwarz' inequality to the sum to show that the sequence $\{S_N\}$ satisfies the Cauchy criterion with respect to the supremum norm. Hence S_N converges uniformly to f.

3. Solve $u_{xx} + u_{yy} = 0$ in the disk $(r < a)$ with the boundary condition $u = \sin^3 \theta$.
   (*Hint:* Use the identity $\sin^3 \theta = \frac{3}{4}\sin\theta - \sin 3\theta$ and apply Poisson's formula.)

4. Derive Poisson's formula for the exterior of the unit disc $(r > 1)$.