APM 351: Differential Equations in Mathematical Physics
Assignment 11, due January 10, 2012

Summary

The **fundamental solution** of the Laplacian in $\mathbb{R}^n$ is given by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi} \log|x| \,, & n = 2\,, \\ -\frac{1}{n(n-2)\omega_n |x|^{n-2}}\,, & n \geq 3\,, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. In three dimensions

$$\Phi(x) = -\frac{1}{4\pi|x|}$$

can be interpreted as the gravitational potential of a point mass, or equivalently, the electrostatic field of a point charge at the origin.

If f is a bounded function on $\mathbb{R}^n$ (where $n \geq 3$) that vanishes outside some ball, then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy\,.$$

is the unique solution of Poisson's equation

$$\Delta u = f\,, \quad x \in \mathbb{R}^n$$

with $u(x) \to 0$ as $|x| \to to\infty$. (There are many other solutions, all of which which grow at infinity.) We say that

$$\Delta \Phi = -\delta$$

in the sense of distributions.

A similar formula holds for Poisson's eqation on a bounded domain in $\mathbb{R}^n$: The unique solution of the Dirichlet problem

$$\Delta u = f\,, \quad \text{for } x \in D\,, \qquad u(x) = g(x)\,, \quad \text{for } x \in \partial D$$

is given by

$$u(y) = \int_D G(y,x) f(y)\, dy + \int_{\partial D} g(y) \nabla_y G(y,x) \cdot \nu(y)\, dS(y)\,.$$

Here, $G(y,x)$ is the **Green's function** of the domain. It is defined by the properties that

- $G(y,x) - \Phi(x,y)$ is smooth and harmonic on D;
- $G(y,x) = 0$ for $y \in \partial D$

for every $x \in D$. We will see that the Green's function is negative and symmmetric, i.e.,

- $G(x, y) = G(y, x)$.

The function defined on the boundary of D by

$$P(x, y) = \nabla_y G(x, y) \cdot \nu(y)$$

is called the **Poisson kernel** associated with D.

The proofs in this section are based on **Green's identities:** For any pair of smooth functions u, v on D, we have

$$\int_D (v\Delta u + \nabla u \cdot \nabla v)\, dx = \int_{\partial D} v \nabla u \cdot \nu(x)\, dS(x)\,, \tag{1}$$

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} (u\nabla v - v\nabla u) \cdot \nu(x)\, dS(x)\,. \tag{2}$$

Assignments:

Read Chapter 7 of Strauss.

1. Suppose that u is a harmonic function in the disk $D = \{r < 1\}$ in two dimensions, and that $u = 3\sin 2\theta + 1$ for $r = 1$. Without computing the solution, find
   (a) the maximum of u on D;
   (b) the value of u at the origin.

2. Find the radial solutions (depending only on $r = |x|$) of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant.
   (*Hint:* Substitute $u(r) = \frac{v(r)}{r}$. Solutions may blow up at $r = 0$.)

3. Let D be an open set with smooth boundary in $\mathbb{R}^3$. Use the divergence theorem to show that the Neumann problem

   $$\Delta u = f \text{ in } D\,, \quad \nabla u \cdot \nu = g \text{ on } \partial D$$

   cannot have a solution unless $\iiint_D f\, dxdydz = \iint_{\partial D} g\, dS$.

4. Consider a homogeneous polynomial in two variables

   $$P(x, y) = a_0 x^k + a_1 x^{k-1} y + \cdots + a_k y^k\,.$$

   (a) Under what conditions on the coefficients is the polynomial harmonic? How many linearly independent harmonic polynomials are there of degree k?

   (b) Write down a basis of the space of harmonic polynomials of degree $k \leq 4$, in both Cartesian and polar coordinates. Identify them as the real (or imaginary) parts of holomorphic functions.