APM 351: Differential Equations in Mathematical Physics
January 11 2012

Summary

By definition, the Green's function of a domain D canbe constructed (for every fixed x) by

$$G(x,y) = \Phi(x-y) - h(y)\,,$$

where Φ is the fundamental solution of Laplace's equation, given by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x|\,, & \text{dimension } n=2\,, \\ -\frac{1}{4\pi|x|}\,, & n=3\,, \end{cases}$$

and h is a harmonic function such that $h(y) = \Phi(x-y)$ whenever y lies in the boundary of D. (Of course, h depends on x as well).

There are only few domains where the Green's function can be computed explicitly. The two most important ones are the upper half-space and the unit ball in $\mathbb{R}^n$. For these, we can use a **reflection principle** to find the harminic function h.

- **Upper half-space:** Let $D = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$. For $x \in D$, we define its reflection at the boundary $\{x_3 = 0\}$ by $\bar{x} = (x_1, x_2, -x_3)$, and set

  $$h(y) = \Phi(\bar{x} - y) = \frac{1}{4\pi}\big((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3+y_3)^2\big)^{-\frac{1}{2}}\,.$$

  Clearly, h is harmonic in y on the entire positive half-space (since $\bar{x}$ lies in the negative half-space). If $y_3 = 0$, then $h(x,y) = \Phi(x-y)$, because in that case $|\bar{x}-y| = |x-y|$. So the Green's function is given by

  $$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|\bar{x}-y|} - \frac{1}{|x-y|}\right)\,,\quad x,y \in D\,.$$

  The **Poisson kernel** for D is given by

  $$K(x,y) = \nabla_y G(x,y)\cdot\nu(y) = \frac{1}{2\pi}\frac{x_3}{|x-y|^3}\,,\qquad x\in D, y\in\partial D\,.$$

- **Unit ball:** Let $D = \{x \in \mathbb{R}^3 \mid |x| < 1\}$. For $x \in D$, we define its reflection at the unit sphere by $\bar{x} = \frac{x}{|x|^2}$. A quick computation shows that

  $$|\bar{x} - \bar{y}|^2 = \frac{|x-y|^2}{|x|^2|y|^2}\,,$$

  in particular, if $y \in \partial D$, then $|\bar{x}-y| = \frac{|x-y|}{|x|}$. For $x \in D$, the function

  $$h(y) = \Phi(|x|\cdot|\bar{x}-y|) = -\frac{1}{4\pi|x|\cdot|\bar{x}-y|}$$

is clearly harmonic in y on D (since $\bar{x}$ lies outside D), and its boundary values agree with those of $\Phi(x-y)$. So the Green's function is given by

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x|\cdot|\bar{x}-y|} - \frac{1}{|x-y|}\right).$$

For the **Poisson kernel**, we obtain (by computing the normal derivative) using that $\nu(y) = y$)

$$K(x,y) = \nabla_y G(x,y)\cdot\nu(y) = \frac{1-|x|^2}{4\pi|x-y|^3}, \qquad x \in D, y \in \partial D.$$

In both cases, we have found a formula for the solution of Poisson's equation

$$\Delta u = f \quad \text{for } x \in D, \qquad u(x) = g(x) \quad \text{for } x \in \partial D.$$

It is given by

$$u(y) = \int_D G(x,y)f(y)\,dy + \int_{\partial D} K(x,y)g(y)\,dS(y).$$

Note that $G < 0$ and $K > 0$, in ageement with the maximum principle.

Assignments

Read Chapter 7 of Strauss.

Hand-in (due Thursday, January 19):

1. Find the Green's function for the Laplacian

   (a) for the positive quadrant in $\mathbb{R}^2$;

   (b) for the upper half of the unit ball in $\mathbb{R}^3$.

   Hint: Recall the fundamental solution of the heat equation, and use reflections.

2. Let D be the unit disc in the plane, and denote by D_+ its intersection with the half-space $y > 0$. Let u be a harmonic function D_+ that is continuous on the closure $\overline{D_+}$. Assume that u vanishes on the flat part of the boundary $\{(x,0) \mid -1 \le x \le 1\}$, and extend it to a function $\tilde{u}$ on the whole disc by odd reflection,

   $$\tilde{u}(x,y) = \begin{cases} u(x,y), & (x,y) \in \overline{D}, y \ge 0 \\ -u(x,y) & (x,y) \in \overline{D}, y \le 0. \end{cases}$$

   Prove that $\tilde{u}$ is harmonic on D, in two ways:

   (a) Show directly that $\Delta\tilde{u}(x,y) = 0$ when $y = 0$.
   Note: You need to assume here that the second derivatives of u are continuous on $\overline{D}_+$.

   (b) Identify $\tilde{u}$ as the solution of a suitable boundary-value problem.
   Hint: What do we know about existence and uniqueness of solutions to this problem?

3. How many linearly independent polynomials of degree k are there in three variables? How many linearly independent *harmonic* polynomials of degree k are there? (*Hint:* Consider the Laplacian as a linear transformation that maps polynomials of degree k to polynomials of degree $k-2$. You may assume that this map is onto.)