APM 351: Differential Equations in Mathematical Physics
Assignment 13, due February 2, 2012

Summary

A **spherical harmonic** of degree ℓ is a function Y on the unit sphere in $\mathbb{R}^n$, such that

$$P(x_1, \dots, x_n) = r^\ell Y(\omega)$$

is a harmonic homogeneous polynomial of degree ℓ. Here $r = |x|$ is the radius, and $\omega = \frac{x}{|x|}$ is the direction vector. Spherical harmonics are useful for solving radially symmetric problems in $\mathbb{R}^n$, such as finding the eigenvalues of the Laplacan on a ball, or the eigenstates of the hydrogen atom in quantum mechanics.

We first construct a basis for the space of harmonic homogeneous polynomials $P(x_1, \dots, x_n)$ of degree ℓ. To do this, we expand such a polynomial in terms of the n-th variable

$$P(x_1, \dots, x_n) = \sum_{k=0}^{\ell} p_k(x_1, \dots, x_{n-1}) x_n^k \,,$$

where each p_k is a homogeneous polynomial of degree $\ell - n$. Setting

$$\Delta P = 0$$

yields the recursion

$$k(k-1)p_k(x_1, \dots, x_{n-1}) + \Delta p_{k-2}(x_1, \dots, x_{n-1}) = 0 \,, \quad k = 2, \dots, \ell.$$

Thus, $P(x_1, \dots, x_n)$ is determined uniquely by specifying two polynomials $p_0(x_1, \dots, x_{n-1})$ of degree ℓ, and $p_1(x_1, \dots, x_{n-1})$ of degree $\ell - 1$.

- $n = 2$ **variables:** We write

  $$P(x, y) = \sum_{k=0}^{\ell} p_k(x) y^k \,,$$

  where $p_k(x) = a_k x^{\ell - k}$. For example, when $\ell = 3$,

  $$\begin{array}{lll} p_0 = x^3, p_1 = 0 & \text{gives} & P = x^3 - 3xy^2 \,, \\ p_0 = 0, p_1 = x^2 & \text{gives} & P = x^2 y - xy^2 \,. \end{array}$$

  In general, choosing (for $\ell \geq 1$)

  $$\begin{array}{lll} p_0 = x^\ell, p_1 = 0 & \text{gives} & P = x^\ell - \frac{\ell(\ell-1)}{2} x^{\ell-2} y^2 + \dots \,, \\ p_0 = 0, p_1 = x^{\ell-1} & \text{gives} & P = x^{\ell-1} y - \frac{(\ell-1)(\ell-2)}{6} x^{\ell-3} y^3 + \cdots \,, \end{array}$$

  and we obtain a basis for the space of homogeneous harmonic polynomials of degree ℓ. For $\ell \geq 1$, this space has dimension 2.

  Alternately, we can use the basis $\{\mathbf{Re}(x + iy)^k, \mathbf{Im}(x + iy)^k\}$. This yields for the space of spherical harmonics of degree ℓ in $\mathbb{R}^2$ the basis $\{\cos(\ell\theta), \sin(\ell\theta)\}$.

- $n = 3$ **variables:** Here,

$$P(x, y, z) = \sum_{k=0}^{\ell} p_k(x, y) z^k \,,$$

where $p_k(x, y)$ has degree $\ell - k$. Again, we get to choose p_0 and p_1, and use the recursion to determine $p_k, \dots, p_\ell$. For example, when $\ell = 3$,

$$\begin{aligned} p_0 = x^3, p_1 = 0 \quad &\text{gives} \quad P = x^3 - 3xz^2 \,, \\ p_0 = x^2y, p_1 = 0 \quad &\text{gives} \quad P = x^2y - yx^2 \,, \\ p_0 = 0, p_1 = x^2 \quad &\text{gives} \quad P = x^2z - xz^2 \,. \end{aligned}$$

The harmonic polynomials of degree ℓ form a vector space of dimension $2\ell + 1$. The spherical harmonics are given by the functions $Y(\theta, \phi) = P(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Notice that the basis we constructed here is different from the basis $\{Y_{\ell,m}(\theta, \phi), m = -\ell, \dots, \ell\}$ that appears in many Physics textbooks.

Assignments

Read Sections 9.1-3 of Strauss.

1. Consider the one-dimensional wave equation $u_{xx} = c^2 u_{xx}$ with initial values given on a surface $\mathcal{S} = \{(x, t) \mid t = \gamma(x)\}$, by

$$u((x, \gamma(x)) = \phi(x) \,, \quad \frac{\partial u}{\partial n} = \Psi(x) \,.$$

If $\mathcal{S}$ is space-like, i.e., $|\gamma'(x)| < \frac{1}{c}$, prove that the initial-value problem has a unique solution. (*Hint:* The solution can be written as $u(x, t) = F(x + ct) + G(x - ct)$.)

2. A **plane wave** is a solution of the wave equation of the form $u(x, t) = f(k \cdot x - ct)$, where f is a C^2-function. Find all the three-dimensional plane waves.

3. Thinking of space-time as $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, let Γ be the diagonal 4×4 matrix with diagonal entries $1, 1, 1, -1$. A **Lorentz transformation** is an invertible matrix that satisfies $L^t \Gamma L = \Gamma$, or equivalently, $L^{-1} = \Gamma L^t \Gamma$.

(a) Prove that Lorentz transformations form a group, i.e., products and inverse of Lorentz transformations are again Lorentz transformations. What can you say about the determinant of L?

(b) Show that L is Lorentz if and only if it preserves the quadratic form $m(x, t) = |x|^2 - t^2$, i.e., $m(L(v)) = m(v)$ for all $v = (x, t) \in \mathbb{R}^4$. The quadratic form m is called the **Lorentz metric**.

(c) If L is a Lorentz transformation, and $U(z) = u(L(z))$, show that

$$u_{tt} - \Delta u = U_{tt} - \Delta U \,,$$

i.e., if u solves the wave equation, so does U.

(d) Explain the meaning of a Lorentz transformation in more geometrical terms. How does m relate to the light cone?